Chapter 7 - Algebraic Methods

Personal notes:

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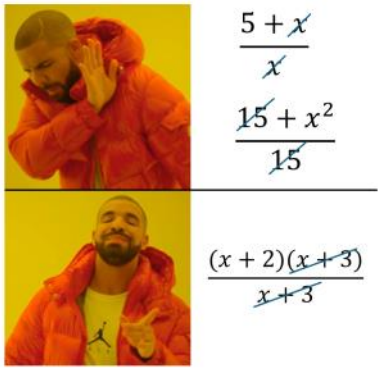
7.1,7.2 - Algebraic fractions & Division

Notes

To simplify algebraic fractions, try the following:

- Factorise as much as you can
- Cancel out factors, not terms!
- Break up the numerators

Algebraic division is the algebra version of long division.



Example

Simplify $\frac{3x^3+2x^2+5x}{x}$

Example

Simplify $\frac{x^2+7x+12}{x+3}$

Example

Simplify $\frac{-2x^2-5x+12}{(x+3)(x+4)}$

Example

Simplify $\frac{-9x^9-6x^6+4x^4-2}{-3x}$

Practice

Simplify these fractions:

a) $\frac{4x^4+5x^2-7x}{x}$

b) $\frac{-4x^2+6x^4-2x}{-2x}$

c) $\frac{x^2+x-12}{x-3}$

d) $\frac{2x^2+7x+6}{(x-5)(x+2)}$

e) $\frac{x^2+6x+8}{3x^2+7x+2}$

7.1,7.2 - Algebraic fractions & Division

Example (Long Division - Number version)

$392 \div 4$

Example (Long Division - Algebraic)

Divide $2x^3 + 3x^2 + x + 6$ by $(x + 2)$.

Example

Given $f(x) = 4x^4 - 17x^2 + 1$. Divide $f(x)$ by $(2x + 1)$ and find the remainder.

7.1,7.2 - Algebraic fractions & Division

Practice Q1

Answer each of the following questions in the form of $(x \pm p)(ax^2 + bx + c)$

a) Divide $x^3 + 6x^2 + 8x + 3$ by $(x + 1)$
b) Divide $x^3 + x^2 - 7x - 15$ by $(x - 3)$
c) Divide $2x^3 - 17x + 3$ by $(x + 3)$

Exam Practice

Given $f(x) = 2x^3 + 3x^2 - 8x + 3$

a) Show that $f(x) = (2x - 1)(ax^2 + bx + c)$, where *a, b* and *c* are constants to be found.
b) Hence factorise $f(x)$ completely. (Hint: all factors are linear, no quadratic brackets)
c) Write down all the real roots of the equation $f(x) = 0$.

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7.3 - The factor theorem

Standard steps of solving quadratics

Solve $f(x) = x^2 + 5x + 6 = 0$

$(x + 2)(x + 3) = 0$

$x + 2 = 0$ or $x + 3 = 0$

$x = -2$ or $x = -3$

- Think about this: What happens if you sub $x = -2$ or $x = -3$ back into $f(x)$?

Notes (Factor Theorem)
If $(x + p)$ is a factor of $f(x)$
then

Example (Number version)
Show that 4 is a factor of 16.

Example
Show that $(x + 1)$ is a factor of $f(x) = x^3 + 2x^2 - 5x - 6$ by

a) Algebraic division
b) Factor Theorem (recommended in exam)

7.3 - The factor theorem

Practice Q1

a) Use factor theorem to show that $(x-4)$ is a factor of $f(x)$, where $f(x)=3x^3-12x^2+6x-24$.

b) Use factor theorem to show that $(x+2)$ is a factor of $f(x)$, where $f(x)=4x^3+4x^2-11x-6$.

c) Use factor theorem to show that $(3x-1)$ is a factor of $f(x)$, where $f(x)=9x^4-18x^3-x^2+2x$.

Exam Practice (May 2016 C2 Q2)

Given $f(x)=6x^3+13x^2-4$

a) Use factor theorem to show that $(x+2)$ is a factor of $f(x)$. **(2 marks)**

b) Factorise $f(x)$ completely. **(4 marks)**

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7.4 - Proof by Deduction

A proof must show all **assumptions** you are using, have a clear **sequential list of steps** that logically follow, and must cover **all possible cases**.
You should usually make a **concluding statement**, e.g. restating the original conjecture that you have proven.

Types of Proof:

- Proof by Deduction
- Proof by Exhaustion
- Disproof by Counter-Example

Proof by Deduction

- It starts with **known facts** or definition, then making logical steps to achieve the proof.
- It is **not** acceptable to start with the conclusion, **because you are assuming the thing you are trying to prove.**

Example

Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5.

Incorrect Proof by a dum dum

Let the lengths be 3,4,5.

Therefore:

$3^2 + 4^2 = 5^2$

$25 = 25$

This satisfies Pythagoras' Theorem, and the numbers are consecutive.

7.4 - Proof by Deduction

Example

Prove that $x^2 + 4x + 5$ is positive for all values of x

Example

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots. Prove k satisties the inequality: $0 \leq k < \frac{8}{9}$.

7.4 - Proof by Deduction

<u>Practice Q1</u>

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

<u>Practice Q2</u>

The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leq k < \frac{12}{25}$.

<u>Practice Q3</u>

Prove that $x^2 + 8x + 20 \geq 4$ for all values of x. (Hint: Look at the previous example)

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7.5 - Proof by Exhaustion and Counter-example

Think & Discuss:
Imagine you are running a school, how can you prove that all the students inside the school are actually students of the school, but not some random strangers?

Proof by exhaustion is breaking the statement into **small cases** and proving **each case** separately.
(Literally exhaustion!!)

Example
Prove by exhaustion that the sum of two consecutive square numbers between 1^2 to 8^2 is an odd number.

Example (Harder proof by deduction question)

a) Prove that for all positive values of x and y:
$\frac{x}{y} + \frac{y}{x} \geq 2$

b) Use a counter-example to show the statement is not true if x or y is not positive.

7.5 - Proof by Exhaustion and Counter-example

Practice Q1

Prove that for all values of x: $(x + 6)^2 \geq 2x + 11$

Practice Q2

Prove that for any positive integers p and q:

$p + q > \sqrt{4pq}$

Hint: $(a - b)^2 = a^2 - 2ab + b^2$

Exam Practice

a) Prove that for all positive values of x and y

$$\sqrt{xy} \leq \frac{x + y}{2}$$

(2 marks)

b) Prove by counter example that this is not always true. **(1 mark)**