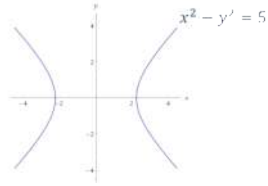
Chapter 8 - Parametric Equations

Personal notes:

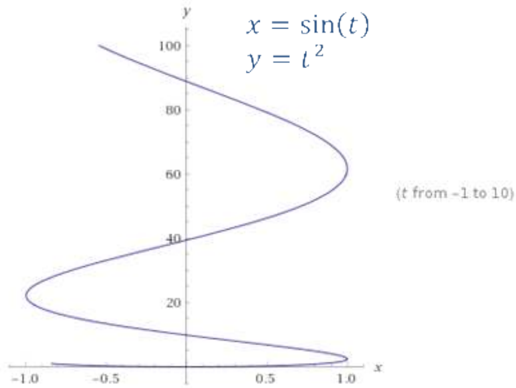
8.1 - Parametric Equations

What are Parametric Equations and what is the point?

- Typically, with two variables x and y, we can relate the two by **a single equation involving just x and y**. This is known as a **Cartesian equation**.



- However, in Mechanics for example, we might want each of the x and y values to be some function of time t, as per this example.
  This would allow us to express the position of a particle at time t as the vector:
  $\begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$
  These are known as **parametric equations**, because **each of x and y are defined in terms of some other variable, known as the parameter** (in this case t).



Conversion from Parametric to Cartesian

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Example

A curve has parametric equations

$x = 2t, \quad y = t^2, \quad -3 < t < 3$

Find a Cartesian equation of the curve in the form of y = f(x)

Domain and Range of Parametrics

- If $x = p(t)$ and $y = q(t)$ can be written as $y = f(x)$, then…

The domain of $f(x)$ can be found using the $x = p(t)$ equation
The range of $f(x)$ can be found using the $y = q(t)$ equation

8.1 - Parametric Equations

Example

A curve has parametric equations

$$x = \ln(t + 3), \quad y = \frac{1}{t + 5}, \quad t > -2$$

a Find a Cartesian equation of the curve of the form $y = f(x)$, $x > k$ where k is a constant to be found.

b Write down the range of $f(x)$.

Practice

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The curve C has parametric equations

$$x = \ln(t + 2), \quad y = \frac{1}{(t + 1)}, \quad t > -1.$$

(c) Find a cartesian equation of the curve C, in the form $y = f(x)$. **(4)**

Also, find the domain and range of y=f(x)

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8.1 - Parametric Equations

Practice

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6. The curve C has parametric equations

$$x = \ln t, \qquad y = t^2 - 2, \qquad t > 0.$$

(*b*) a cartesian equation of C.

(3)

Find the domain and range of y=f(x)

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8.2 - Parametric with Trig Identities

Example

A curve has parametric equations $x = \sin t + 2, \quad y = \cos t - 3, \quad t \in \mathbb{R}$

a Show that a Cartesian equation of the curve is $(x - 2)^2 + (y + 3)^2 = 1$.

b Hence sketch the curve.

Example

A curve is defined by the parametric equations

$$x = \sin t, \quad y = \sin 2t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

a Find a Cartesian equation of the curve in the form

$$y = \mathrm{f}(x), \quad -k \leqslant x \leqslant k$$

stating the value of the constant k.

b Write down the range of $\mathrm{f}(x)$.

8.2 - Parametric with Trig Identities

Practice

C4 June 2013

4. A curve C has parametric equations

$$x = 2\sin t, \qquad y = 1 - \cos 2t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

(*b*) Find a cartesian equation for C in the form

$$y = f(x), \qquad -k \leq x \leq k,$$

stating the value of the constant k.

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8.2 - Parametric with Trig Identities

Practice

A curve C has parametric equations

$$x = \cot t + 2 \quad y = \operatorname{cosec}^2 t - 2, \quad 0 < t < \pi$$

a Find the equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined.

b Hence, sketch the curve.

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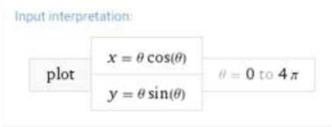
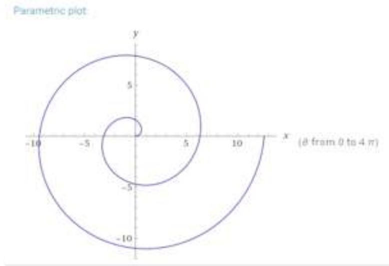
8.3 - Curve Sketching

The strategy for sketching parametric curves is to convert into a Cartesian equation, and hope this is a form we recognise (e.g. quadratic or equation of circle) to appropriately sketch.
However, some parametric equations can't easily be turned into Cartesian form:

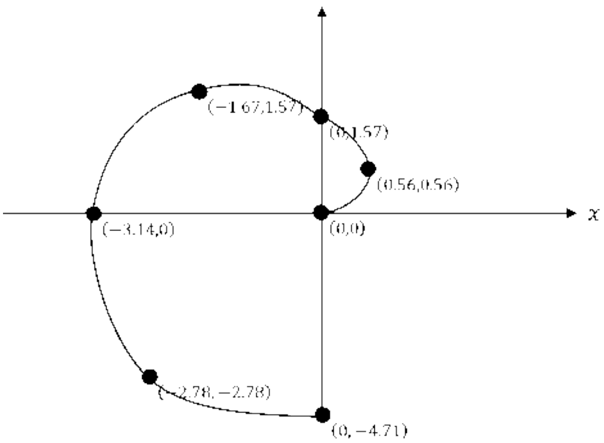




These parametric equations in Cartesian form would be $\sqrt{x^2+y^2} = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$; this would obviously be incredibly hard to sketch!

Set up a table of values of x and y

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\boldsymbol{x}$	0	0.56	0	-1.67	-3.14	-2.78	0	3.89	6.28
$\boldsymbol{y}$	0	0.56	1.57	1.67	0	-2.78	-4.71	-3.89	0



8.3 - Curve Sketching

Example

Draw the curve given by the parametric equations $x = 2t$, $y = t^2$, for $-1 \leqslant t \leqslant 5$.

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8.4 - Points of Intersection

Points of Intersection = Simultaneous Equations!
The key is to find the value of *t* first.

Example

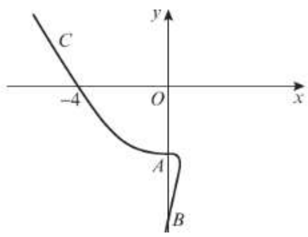
A curve is given parametrically by the equations $x = t^2$, $y = 4t$. The line $x + y + 4 = 0$ meets the curve at A. Find the coordinates of A.

Example

The diagram shows a curve C with parametric equations $x = at^2 + t$, $y = a(t^3 + 8)$, $t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4, 0)$,

a find the value of a

b find the coordinates of the points A and B where the curve crosses the y-axis.



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8.4 - Points of Intersection

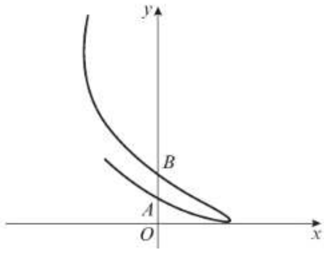
Example

The diagram shows a curve C with parametric equations

$$x = \cos t + \sin t, \quad y = \left(t - \frac{\pi}{6}\right)^2, \quad -\frac{\pi}{2} < t < \frac{4\pi}{3}$$

a Find the point where the curve intersects the line $y = \pi^2$.

b Find the coordinates of the points A and B where the curve cuts the y-axis.



Practice

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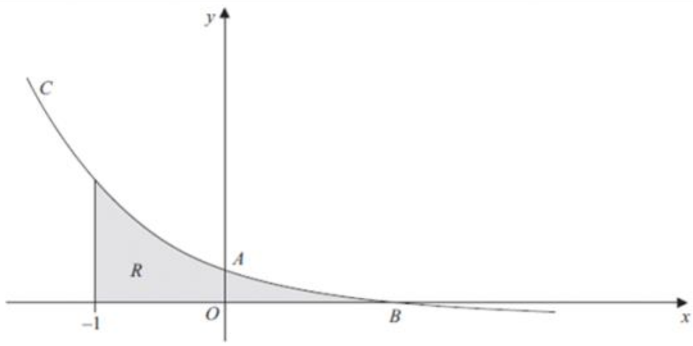


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates (0, 3).

(2)

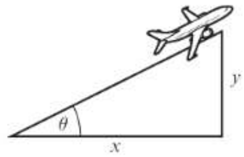
(*b*) Find the x-coordinate of the point B.

(2)

8.5 - Modelling with parametric

- Parametric equations are frequently used in mechanics, particularly where the (x, y) position (the Cartesian variables) depends on time t (the parameter).
- Always link the information given to the variables in the equations. Think of which variable is the question asking.

Example



A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \quad y = (v\sin\theta)t \text{ m}, \quad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 600 m horizontally, it has climbed 120 m

a Find the angle of elevation, θ.

Given that the plane's speed is 50 m s^{-1},

b find the parametric equations for the plane's motion

c find the vertical height of the plane after 10 seconds

d show that the plane's motion is a straight line

e explain why the domain of t, $t > 0$, is not realistic.

8.5 - Modelling with parametric

<u>Example</u>

The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t, \quad y = 12\sin\left(10t - \frac{\pi}{3}\right), \quad t \geqslant 0$$

where x and y are measured in metres.

a Find the coordinates of the figure skater at the beginning of his motion.

b Find the coordinates of the point where the figure skater intersects his own path.

c Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d Determine how long it takes the figure skater to complete one complete figure-of-eight motion.