Chapter 7 - Trigonometry and Modelling

Personal notes:

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7.1 - Compound Angle Formulae

Starter

1. Solve the following questions in the interval $0 \leq x \leq 360°$

a) $\sin(x + 50) = -0.9$

b) $2\sin^2 x - \sin x - 3 = 0$

2. Prove the following.

a) $\cos x + \sin x \tan x \equiv \sec x$

b) $\dfrac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \sin^2 x$

Compound Angle Formulae

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7.1 - Compound Angle Formulae

Example (Ex. 7A Q9a,e)

Express the following as a single sine, cosine or tangent:

a) $sin 15° \, cos 20° + cos 15° \, sin 20°$

e) $\cos 2\theta \cos \theta + \sin 2\theta \, sin\theta$

Example/Practice

Edexcel C3 Jan 2012 Q8

(a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

7.2 - Using the compound angle formulae

- The compound angle formulae can be used to find **<u>exact values</u>** of trig functions of different angles.

<u>Example</u>

Using a suitable angle formulae, show that $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

<u>Example</u>

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$ and B is obtuse, find the value of:

a $\cos(A - B)$ **b** $\tan(A + B)$ **c** $\text{cosec}(A - B)$

7.2 - Using the compound angle formulae

<u>Challenging Question</u>

Given that

$$2\cos(x+50)^\circ = \sin(x+40)^\circ.$$

(*a*) Show, without using a calculator, that

$$\tan x^\circ = \frac{1}{3}\tan 40^\circ.$$

(*b*) Hence solve, for $0 \leq \theta < 360$,

$$2\cos(2\theta+50)^\circ = \sin(2\theta+40)^\circ,$$

giving your answers to 1 decimal place.

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7.3 - Double Angle Formulae

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Example

Use the double-angle formulae to write each of the following as a single trigonometric ratio.

a $\cos^2 50° - \sin^2 50°$ **b** $\dfrac{2\tan\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{6}}$ **c** $\dfrac{4\sin 70°}{\sec 70°}$

Practice (Ex. 7C Q4)

4 Write each of the following expressions as a single trigonometric ratio.

a $2\sin 10° \cos 10°$ **b** $1 - 2\sin^2 25°$ **c** $\cos^2 40° - \sin^2 40°$

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7.3 - Double Angle Formulae

Example

Given that $x = 3\sin\theta$ and $y = 3 - 4\cos 2\theta$, eliminate θ and express y in terms of x.

Practice (Ex. 7C Q9)

9 Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta$, $y = 1 - \cos 2\theta$

b $x = \tan\theta$, $y = \cot 2\theta$

Example

Given that $\cos x = \frac{3}{4}$, and that $180° < x < 360°$, find the exact value of:

a $\sin 2x$

b $\tan 2x$

7.4 - Solving Trigonometry Equations

Compound Angle Formulae

- $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ $\quad \sin(A-B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ $\quad \cos(A-B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad \tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Double Angle Formulae

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

Example

Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0 \leqslant \theta \leqslant 360°$. Round your answer to 1 decimal place.

Example

Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leqslant x \leqslant 360°$.

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7.4 - Solving Trigonometry Equations

Practice (Ex. 7D Q5c,e,g)

Solve the following equations in the given intervals

c $3\cos 2\theta = 2\cos^2\theta,\ 0 \leqslant \theta < 360°$

e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0,\ 0 \leqslant \theta < 720°$

g $2\sin\theta = \sec\theta,\ 0 \leqslant \theta \leqslant 2\pi$

7.4 - Solving Trigonometry Equations

Example

a By expanding $\sin(2A + A)$ show that $\sin 3A \equiv 3\sin A - 4\sin^3 A$.

b Hence, or otherwise, for $0 < \theta < 2\pi$, solve $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$ giving your answers in terms of π.

Extension

(a) Solve, for $0 \leqslant \theta < 2\pi$,

$$\sin\left(\frac{\pi}{3} - \theta\right) = \frac{1}{\sqrt{3}}\cos\theta .$$

(5)

(b) Find the value of x for which

$$\arcsin(1 - 2x) = \frac{\pi}{3} - \arcsin x, \qquad 0 < x < 0.5$$

[$\arcsin x$ is an alternative notation for $\sin^{-1} x$]

(7)

7.5 (1) - Simplifying $a\cos x \pm b\sin x$

- **For positive values of a and b,**
  - **$a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$**
  - **$a\cos x \pm b\sin x$ can be expressed in the form $R\cos(x \mp \alpha)$**

  with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

  where $R\cos\alpha = a$ and $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$.

<u>Example</u>
Put $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$ giving α in degrees to 1dp.

<u>Practice</u>
Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ giving α in terms of degrees

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7.5 (1) - Simplifying $a\cos x \pm b\sin x$

<u>Example</u>

a) Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ giving α in terms of degrees
b) Sketch the graph of $y = 2\cos\theta + 5\sin\theta$
c) Hence, solve for $0 < \theta \leq 360°$, the equation $2\cos\theta + 5\sin\theta = 3$

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7.5 (2) - Maxima and Minima

<u>Example</u> **Exam Favourite

Express $12\cos\theta + 5\sin\theta$ in the form of $R\cos(\theta - \alpha)$ and hence find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

<u>Practice</u>

Find the maximum value of the expression and determine the smallest positive value of θ (in degrees) at which it occurs.

Expression	Maximum Value	(Smallest) θ at max
$20\sin\theta$		
$5 - 10\sin\theta$		
$3\cos(\theta + 20°)$		
$\dfrac{2}{10 + 3\sin(\theta - 30)}$		

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7.5 (2) - Maxima and Minima

Exam Practice (Edexcel C3 Jan 2013 Q4)

4. (*a*) Express 6 cos θ + 8 sin θ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*)

$$p(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Calculate

(i) the maximum value of $p(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)

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7.6 - Proving Trigonometric identities

Key skill: Apply appropriate compound angle formulae and double angle formulae.

<u>Structure of Proof</u>

- Start either by the LHS or the RHS
- Manipulate the expression and write in the form as the RHS
- Write " = LHS/RHS" to finish off.

<u>Example</u>

Prove the identity $\tan 2\theta \equiv \dfrac{2}{\cot\theta - \tan\theta}$

<u>Example</u>

Prove that $\sqrt{3}\cos 4\theta + \sin 4\theta \equiv 2\cos\left(4\theta - \frac{\pi}{6}\right)$.

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7.6 - Proving Trigonometric identities

<u>Extension</u>

Edexcel C3 June 2015 Q8

(a) Prove that

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}, \qquad A \neq \frac{(2n+1)\pi}{4}, \ n \in \mathbb{Z}$$

(5)

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$\sec 2\theta + \tan 2\theta = \frac{1}{2}$$

Give your answers to 3 decimal places.

(4)

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7.7 - Modelling with trigonometric functions

- In trigonometrical modelling questions, you will often have to write the model using $Rsin(x \pm \alpha)$ or $Rcos\ (x \pm \alpha)$ to find maximum and minimum values.

<u>Example</u>

The cabin pressure, P, in pounds per square inch (psi) on an aeroplane at cruising altitude can be modelled by the equation $P = 11.5 - 0.5\sin(t - 2)$, where t is the time in hours since the cruising altitude was first reached, and angles are measured in radians.

a State the maximum and the minimum cabin pressure.

b Find the time after reaching cruising altitude that the cabin first reaches a maximum pressure.

c Calculate the cabin pressure after 5 hours at a cruising altitude.

d Find all the times during the first 8 hours of cruising that the cabin pressure would be exactly 11.3 psi.

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7.7 - Modelling with trigonometric functions

<u>Example</u>

[June 2013 (Withdrawn) Q8]

(a) Express $9\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the exact value of R and give the value of α to 4 decimal places. **(3)**

(b) (i) State the maximum value of $9\cos\theta - 2\sin\theta$

(ii) Find the value of θ, for $0 < \theta < 2\pi$, at which this maximum occurs. **(3)**

Ruth models the height H above the ground of a passenger on a Ferris wheel by the equation

$$H = 10 - 9\cos\left(\frac{\pi t}{5}\right) + 2\sin\left(\frac{\pi t}{5}\right)$$

where H is measured in metres and t is the time in minutes after the wheel starts turning.

(c) Calculate the maximum value of H predicted by this model, and the value of t, when this maximum first occurs. Give your answers to 2 decimal places. **(4)**

(d) Determine the time for the Ferris wheel to complete two revolutions. **(2)**

7.7 - Modelling with trigonometric functions

Practice

[June 2010 Q7] 2. (*a*) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 4 decimal places. **(3)**

(*b*) (i) Find the maximum value of $2\sin\theta - 1.5\cos\theta$.

(ii) Find the value of θ, for $0 \leq \theta < \pi$, at which this maximum occurs. **(3)**

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right), \quad 0 \leq t < 12,$$

where t hours is the number of hours after midday.

(*c*) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs. **(3)**

(*d*) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres. **(6)**

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