Chapter 10 - Numerical methods

Personal notes:

10.1 - Locating roots

Notes

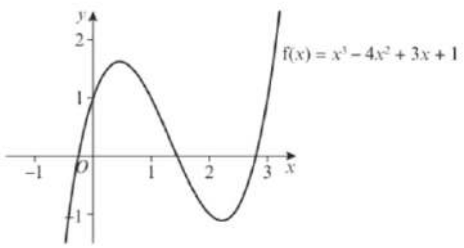
- Recall: The roots of $y = f(x)$ are the values of x such that $f(x) = 0$.
  Graphically, it is where the graph crosses the $x - axis$.

- If the function $f(x)$ is *continuous* and there is at least one root between the interval $[a, b]$, then ____
  (as the graph crosses the $x - axis$). We could use the logic of "sign change" to determine the existence of roots between the interval.

Example

The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = x^3 - 4x^2 + 3x + 1$.

a) Explain how the graph shows that $f(x)$ has a root between $x = 2$ and $x = 3$.
b) Show that $f(x)$ has a root between $x = 1.4$ and $x = 1.5$.



Practice

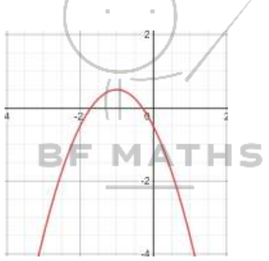
Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$

Investigation 1

Sketch the graph of $f(x) = \frac{1}{x}$.
Work out the value of $f(1)$ and $f(-1)$.
Could you conclude there is a root between $x = -1$ and $x = 1$?
Why or why not?

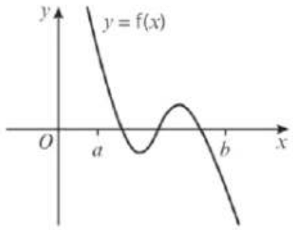
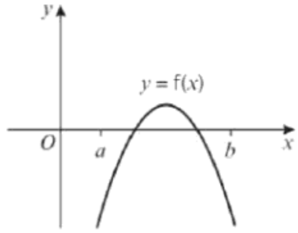
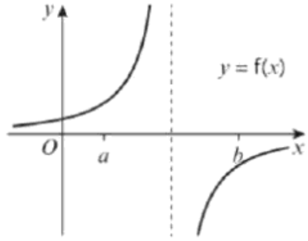
Investigation 2

Given $f(x) = -(x + 1)^2 + 0.5$ and its graph is attached.
Work out $f(-2)$ and $f(0)$.
Could you conclude there is no root between [-2 , 0] ?



10.1 - Locating roots

<u>Potential fallacy of sign-change method to locate a root</u>

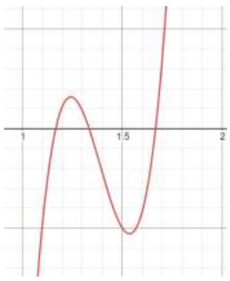
Graphical Representation			
Problem	Sign changes between [a,b]…but	No sign change between [a,b]	Sign changes between [a,b]…but

<u>Example</u>

The graph of the function $f(x) = 54x^3 - 225x^2 + 309x - 140$ is shown in the diagram. A student observes that $f(1.1)$ and $f(1.6)$ are both negative and states that $f(x)$ has no roots in the interval (1.1, 1.6).

a) Explain by reference to the diagram why the student is incorrect.

b) Calculate $f(1.3)$ and $f(1.5)$ and use your answer to explain why there are at least 3 roots in the interval $1.1 < x < 1.7$.



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10.1 - Locating roots

Practice

a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain, with the aid of your diagram that the function $f(x) = \ln x - \frac{1}{x}$ has only one root. (Hint: consider the simultaneous equation of $y = \ln x$ and $y = \frac{1}{x}$)

b) Show that this root lies in the interval $1.7 < x < 1.8$.

c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places. (Hint: consider lower bound and upper bound of 1.763)

Exam Practice

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$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. **(3)**

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10.2 - Iteration

<u>Notes</u>

- Sign-change method helps us to *approximate the location of roots* (but unable to work out the exact root).
- we could also use ***<u>iteration</u>*** to *approximate the actual root* (for which $f(x) = 0$).

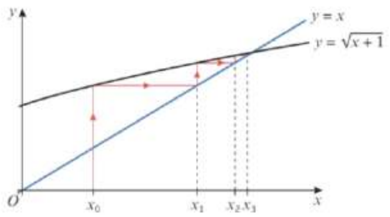
- To solve an equation of the form $f(x) = 0$ by an iterative method,

- Each iteration could either be ________________________________.

  - Two scenarios for _______________iteration:

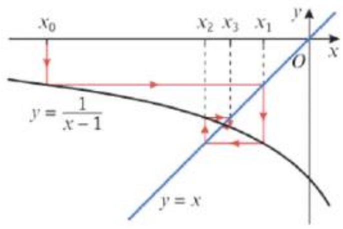
<u>Illustration (Converging - staircase)</u>

$f(x) = x^2 - x - 1$ can be rearranged and produce the iterative formula $x_{n+1} = \sqrt{x_n + 1}$ when $f(x) = 0$.
Let $x_0 = 0.5$



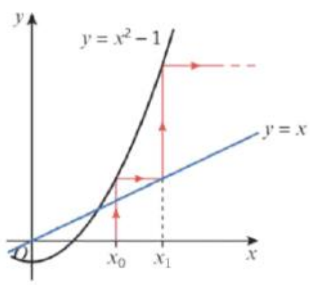
<u>Illustration (Converging - cobweb)</u>

$f(x) = x^2 - x - 1$ can be rearranged and produce the iterative formula $x_{n+1} = \frac{1}{x_n - 1}$ when $f(x) = 0$.
Let $x_0 = -2$



<u>Illustration (Diverging)</u>

$f(x) = x^2 - x - 1$ can be rearranged and produce the iterative formula $x_{n+1} = (x_n)^2 - 1$ when $f(x) = 0$.
Let $x_0 = 2$



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10.2 - Iteration

<u>Example</u>

$f(x) = x^2 - 4x + 1$

a) Show that the equation $f(x) = 0$ can be written as $x = 4 - \frac{1}{x}, x =\neq 0$.

Given $f(x)$ has a root, α, in the interval $3 < x < 4$.

b) $x_{n+1} = 4 - \frac{1}{x_n}$ with $x_0 = 3$ to find the value of x_1, x_2 and x_3.

<u>Example</u>

Given $\mathrm{f}(x) = x^3 - 3x^2 - 2x + 5$,

a) Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$.

b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking:
i) $x_0 = 1.5$ ii) $x_0 = 4$

<u>Practice</u>

Given $f(x) = x^2 - 5x - 3$

a) Show that $f(x) = 0$ can be written as :

i) $x = \sqrt{5x + 3}$ ii) $x = \frac{x^2 - 3}{5}$

b) Let $x_0 = 5$. Work out x_1, x_2 and x_3 to 5 decimal places.

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10.2 - Iteration

Exam Practice

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$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

10.3 - Newton-Raphson method

Notes

- In addition to *iteration*, we could also use <u>*Newton-Raphson formula*</u> to approximate the numerical value of the roots of $y = f(x)$ when $f(x) = 0$.

- The Newton-Raphson formula is :

- The formula will fail when ____________________________ because

Example

Given $x = \cos x$, use Newton-Raphson procedure three times to find a third approximation to α with $x_0 = 0.5$.

Practice

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$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(c) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.
(5)

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$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(c) Taking 1.4 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α, giving your answer to 3 decimal places.
(5)

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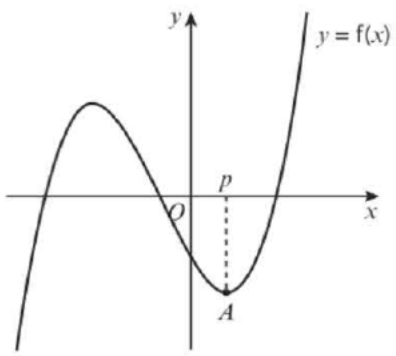
10.3 - Newton-Raphson method

<u>Practice</u>

The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = x^3 + 2x^2 - 5x - 4$.
The point A, with x-coordinate p, is a stationary point on the curve. The equation $f(x) = 0$ has a root, α, in the interval $1.8 < \alpha < 1.9$.

a) Explain why $x_0 = p$ is not suitable to use as a first approximation to α when applying the Newton-Raphson method to $f(x)$.
b) Using $x_0 = 2$ as a first approximation to α, apply the Newton-Raphson procedure twice to find a second approximation to α, giving your answer to 3 decimal places. (Hint: See notes section in previous page)
c) By considering the change of sign in $f(x)$ over an appropriate interval, show that your answer to part **b** is accurate to 3 decimal places.



10.4 - Application to modelling

Notes

- You can apply the same skills to estimate solutions real-life situation models.

Example

The price of a car in £s, x years after purchase, is modelled by the function

$f(x) = 15\ 000\ (0.85)^x - 1000 \sin x\,, \qquad x > 0$

a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.
b) Show that $f(x)$ has a root between 19 and 20.
c) Find $f'(x)$
d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.
e) Criticise this model with respect to the value of the car as it gets older.

10.4 - Application to modelling

Practice

An astronomer is studying the motion of a planet moving along an elliptical orbit. She formulates the following model relating the angle moved at a given time, *E* radians, to the angle the planet would have moved if it had been travelling on a circular path, *M* radians:

$$M = E - 0.1 \sin E \,, E \geq 0$$

In order to predict the position of the planet at a particular time, the astronomer needs to find the value of *E* when $M = \frac{\pi}{6}$

a) Show that this value of *E* is a root of the functions $f(x) = x - 0.1 \sin x - k$ where k is a constant to be determined.
b) Taking 0.6 as a first approximation, apply the Newton-Raphson procedure once to obtain a second approximation for the value of *E* when $M = \frac{\pi}{6}$, to 3 decimal places.
c) By considering a change of sign on a suitable interval of $f(x)$, show that your answer to part **b** is correct to 3 decimal places.