Chapter 14 - Exponentials and Logarithms

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Personal notes:

14.1 - Intro to exponential functions

Notes

- Quadratic function: $f(x) = ax^2 + bx + c$
- Cubic function: $f(x) = ax^3 + bx^2 + cx + d$
- Exponential function: $f(x) = a^x$

Graph of $y = a^x$, where $a > 1$

Graph of $y = a^x$, where $a < 1$

Example/Practice

The graph of $f(x) = ka^x$ passes through the points $(1,6)$ and $(4,48)$. Find the values of the constants k and a.

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14.2 - $y = e^x$

<u>The exponential number **e**</u>

Once upon a time about 300 years ago, there was a little fella from Switzerland called ***Euler.*** His parents divorced when he was only 10 and he was sent to live with his grandma. However, little did they know about this little Euler is a genius. He entered the University of Basel at the age of 13. 3 years later, at the age of 16, he received a Master of Philosophy. His Saturday School Maths Teacher Bernoulli (also another very famous mathematician) discovered Euler's talent in mathematics and he was invited to work alongside Bernoulli's sons in Russian Academy of Sciences in Saint Petersburg. Euler worked in almost all areas of Mathematics, e.g. geometry, algebra, number theory, calculus…etc. One of his famous findings is the Exponential Number ***e***. All people are created equal. As clever mathematician as Euler, he had an eyesight problem throughout his mathematical career and his right eye eventually became fully blind and soon affected his left eye too. At the age of 76, he suffered from a brain haemorrhage and he passed away. However, a lot of people will remember him and thank for his great contribution in the mathematics world as most of his work is still being used in nowadays advanced mathematics study.

- The exponential number **e** is similar to $\boldsymbol{\pi}$, where both numbers have non-ending decimal values.

- $e \approx 2.71828 \ldots$

- Euler has found out that when e is the base of the exponential function ($y = e^x$), then gradient at any point of the graph is the same as the y-value of that point.

- For all values of x:
  If $f(x) = e^x$, then $f'(x) =$

<u>Example</u>

a) Differentiate e^{4x}
b) Differentiate $3e^{2x}$

14.2 - $y = e^x$

Practice Q1

Differentiate the following with respect to x.

a) e^{3x} b) $8e^{1.5x}$ c) $e^{5x} + 3e^{-x}$ d) $e^{2x}(e^x + 1)$

Example

Sketch the graph of the following functions and state the equation of asymptote:

a) $y = e^x$
b) $y = e^{2x}$
c) $y = 10e^{-x}$
d) $y = 3 + 4e^{\frac{1}{2}x}$

Practice Q2

Sketch the graphs of:

a) $y = e^{x+1}$ b) $y = 2e^x - 3$ c) $y = 100e^{-x} + 10$

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14.3 - Exponential modelling

Notes

- *Linear model* is good at representing direct proportional relationship between two variables. However, in some real-life situations, some variables increase/decrease much more vigorously, e.g. growth of bacteria over time, decay of radioactive particles over time…etc

- Say we start with 2 mg of bacteria, and it grows 10% over second. Then the multiplier would be 1.1 - so…
  When t = 1 (after 1 second), there would be 2 x 1.1 = 2×1.1^1 mg of bacterial
  When t = 2 (after 2 seconds), there would be 2 x 1.1 x 1.1 = 2×1.1^2 mg of bacteria
  When t = 3 (after 3 seconds), there would be 2 x 1.1 x 1.1 = 2×1.1^3 mg of bacteria

- In summary, we can represent it as an *exponential model,* and write it as $P = 2 \times 1.1^t$.
- More generally, $P = ab^t$, where a is the initial value.

Example

The density of a pesticide in a given section of field, $P\ mg/m^2$, can be modelled by the equation $P = 160e^{-0.006t}$, where *t* is the time in days since the pesticide was first applied.

a) Use this model to estimate the density of pesticide after 15 days.
b) Interpret the meaning of the value *160* in this model.
c) Show that $\frac{dP}{dt} = kP$, where *k* is a constant and state the value of *k*.
d) Interpret the significance of the sign of your answer to part **c**.
e) Sketch the graph of *P* against *t*.

14.3 - Exponential modelling

Practice Q1

On Earth, the atmospheric pressure, *p,* in bars can be modelled approximately by the formula $p = e^{-0.13h}$ where *h* is the height above sea level in kilometres.

a) Use this model to estimate the pressure at the top of Mount Rainier, which has an altitude of 4.394km.

b) Show that $\frac{dp}{dh} = kp$, where *k* is a constant to be found.

c) Interpret the significance of the sign of *k* in part **b**.

d) This model predicts that that atmospheric pressure change by $s\%$ for every kilometre gained in height. Calculate the value of *s*.

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14.4 - Intro to logarithms

Ever wonder how to work out these questions (apart from trial and error?)

$2^x = 65536$
$3^x = 531441$

Notes
The inverse of exponential function is ***logarithms (log)***.

Example
Rewrite the following exponential equations using logarithms:

a) $4^{12} = 16777216$
b) $(0.3)^4 = 0.0081$

Example
Rewrite the following using a power

a) $\log_2 1024 = 10$
b) $\log_{0.5} 64 = -6$

Practice Q1
Rewrite the following in logarithms/power:

a) $3^{-5} = \frac{1}{243}$

b) $\log_5 25 = 2$

c) $10^6 = 1000000$

d) $\log_{0.25} 16 = -2$

14.4 - Intro to logarithms

<u>Example</u>

Solve the following equations:

a) $\log_7 x = 1$
b) $\log_3(4x + 1) = 4$

<u>Practice Q2</u>

Solve the following equations:

a) $\log_2(x - 1) = 3$
b) $\log_x 2x = 2$

14.5 - Laws of logarithms

Notes
Like normal algebra, there are some laws/rules to follow when simplifying logarithms:

- Base Law (BL)

- Multiplication Law (ML)

- Division Law (DL)

- Power Law (PL)

Example
Simplify $\log_6 x + \log_6 y$

Example
Simplify $2\log_5 3 - 3\log_5 2$

Example
Write the following expression in terms of $\log x$, $\log y$ and $\log z$:
$\log(x^2yz^3)$

Practice Q1
Simplify the following expressions:
a) $\log_2 15 - \log_2 3$
b) $\log_{10} 3 - 4\log_{10}(\frac{1}{2})$
c) $2\log 2 - (\log 5 + \log 8)$
d) $\log_a(\frac{x\sqrt{y}}{z})$

14.6 - Solving equations with logarithms

Notes
You will need to solve different equations involving logarithms.

The trick is to use *the appropriate log laws to simplify to a single log*.

Example
Solve the following equation
$\log 4 + 2\log x = 2$

Example
Solve $5^{4x-1} = 60$ and give your answer in 3s.f.

Practice Q1
Solve the following equations, give your answers in 3s.f. if necessary:

a) $4^{2x} = 100$
b) $2^{3-2x} = 88$
c) $\log_2 3 + \log_2 x = 2$
d) $2\log_9(x+1) = 2\log_9(2x-3) + 1$

14.6 - Solving equations with logarithms

Example
Solve $5^{2x} - 12(5^x) + 20 = 0$

Example
Solve $3^{2x+1} - 17(3^x) + 10 = 0$

Practice Q2
Solve the following equations, give your answers in 3s.f. if necessary:

a) $2^{2x} - 6(2^x) + 5 = 0$
b) $7^{2x} - 7^{x+1} = -12$
c) $4\left(3^{(2x+1)}\right) + 17(3^x) - 7 = 0$

Challenge
Solve $2(4^x) + 3(4^{-x}) = 7$

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14.7 - Natural logarithms

Notes

- By default, logarithm has base 10. When a logarithm has base *e*, it is known as *natural logarithm* (denoted as *ln*).
- The reverse of e^x is *ln* and the reverse of *ln* is e^x.

$e^{\ln x}$ *and* $\ln(e^x)$

- $y = e^{\ln x}$
- $y = \ln(e^x)$

Summary

To get rid of e in the e^x,

To get rid of ln in the $\ln x$,

Graph of $y = \ln x$

Example
Solve $e^x = 5$

Example
Solve $\ln x = 3$

Example
Solve $e^{2x+3} = 7$

Example
Solve $2\ln x + 1 = 5$

14.7 - Natural logarithms

Practice Q1

Solve the following equations and give your answers in exact value:

a) $e^x = 6$

b) $e^{-x+3} = 20$

c) $\ln(18 - x) = \frac{1}{2}$

d) $2\ln(3x - 5) = 7$

Example

Solve $e^{2x} + 5e^x = 14$

Practice Q2

Solve the following equations and give your answers in exact value:

a) $e^x - 3e^{\frac{x}{2}} + 2 = 0$

b) $e^x - 5 + 4e^{-x} = 0$

14.7 - Natural logarithms

Exam Practice/Challenge

Solve $3^x e^{4x-1} = 5$, giving your answer in the form $\frac{a+\ln b}{c+\ln d}$. **(5 marks)**

(Hint: Take ln on both sides to start)

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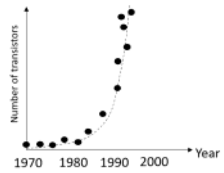
14.8 - Modelling with logarithms

Starter Q1
Apply log to both sides of the equation $y = ax^n$ and make $\log y$ the subject.

Starter Q2
Apply log to both sides of the equation $y = ab^x$ and make $\log y$ the subject.

Notes
Sometimes, when plotting the raw data collected in scientific experiments, they do not make a neat graph, or at times rather useless.

By "log-ing" the data, a better straight line graph can be formed.



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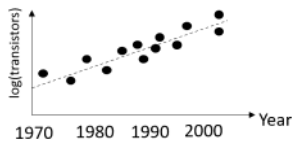


14.8 - Modelling with logarithms

Example

The table below gives the rank (by size) and population of the UK's largest cities and districts (London is ranked number 1 but has been excluded as an outlier).

City	Birmingham	Leeds	Glasgow	Sheffield	Bradford
Rank, R	2	3	4	5	6
Population, P (2 s.f.)	1 000 000	730 000	620 000	530 000	480 000

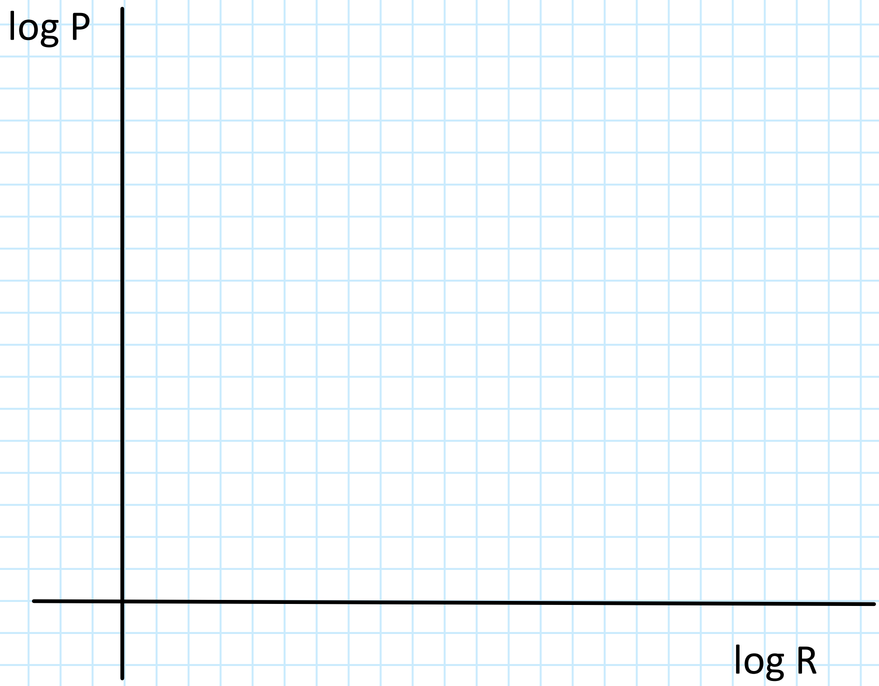
The relationship between the rank and population can be modelled by the formula

$R = aP^n$ where a and n are constants.

a Draw a table giving values of log R and log P to 2 decimal places.

b Plot a graph of log R against log P using the values from your table and draw a line of best fit.

c Use your graph to estimate the values of a and n to two significant figures.



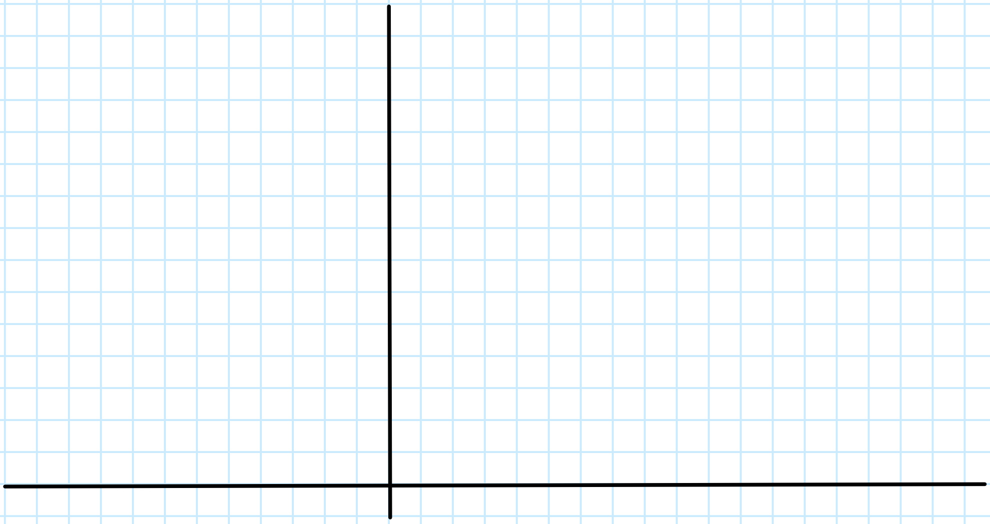
14.8 - Modelling with logarithms

Practice Q1

Kleiber's law is an empirical law in biology which connects the mass of an animal, *m*, to its resting metabolic rate, *R*. The law follows the form $R = am^b$, where *a* and *b* are constants. The table below contains data on five animals

Animal	Mouse	Guinea pig	Rabbit	Goat	Cow
Mass, *m* (kg)	0.03	0.408	4.19	34.6	650
Metabolic rate *R* (kcal per day)	4.2	32.3	195	760	7637

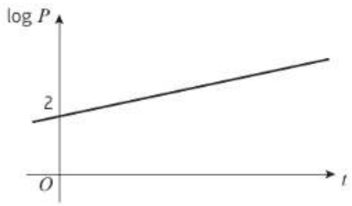
a) Draw a table giving values of $\log m$ and $\log R$, correct to 2 decimal places.
b) Plot a graph of $\log R$ against $\log m$ using the values from your table and draw a line of best fit.
c) Use your graph to estimate the values of *a* and *b* to two significant figures.
d) Use $a = 60$ and $b = 0.75$, estimate the resting metabolic rate of a human male with a mass of 80kg.



14.8 - Modelling with logarithms

Example

The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at $(0, 2)$ as shown.



A scientist suggests that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

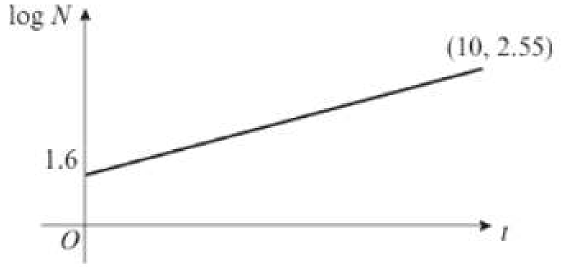
a Write down an equation for the line.

b Using your answer to part **a** or otherwise, find the values of a and b, giving them to 3 significant figures where necessary.

c Interpret the meaning of the constant a in this model.

Practice Q2

A scientist is modelling the number of people, *N*, who have fallen sick with a virus after *t* days.



From the graph, the scientist suggests that the number of sick people can be modelled by the equation $N = ab^t$, where *a* and *b* are constants are to be found.

The graph passes through the points $(0, 1.6)$ and $(10, 2.55)$.

a) Write down the equation of the line.
b) Using your answer to part **a** or otherwise, find the values of *a* and *b*, giving them to 2 significant figures.
c) Interpret the meaning of the constant *a* in this model.
d) Use your model to predict the number of sick people after 30 days. Give one reason why this might be an overestimate.