

Chapter 12 - Differentiation

12.1 - Intro to differentiation - Pg. 2

12.3 - Differentiating x^n - Pg. 3

12.4 - Differentiating quadratics - Pg. 4 - 5

12.5 - Differentiating harder functions - Pg. 6 - 7

12.6 - Gradients, tangents and normal - Pg. 8 - 9

12.7 - Increasing/decreasing functions - Pg 10 - 11

12.8 - Second order derivatives - Pg. 12

12.9 - Stationary points - Pg. 13 - 15

12.10 - Sketching gradient functions - Pg. 16 - 17

12.11 - Modelling with differentiation - Pg. 18 - 21

12.2 - Differentiation by first principles - Pg. 22 - 24

Personal notes:



12.1 - Intro to differentiation

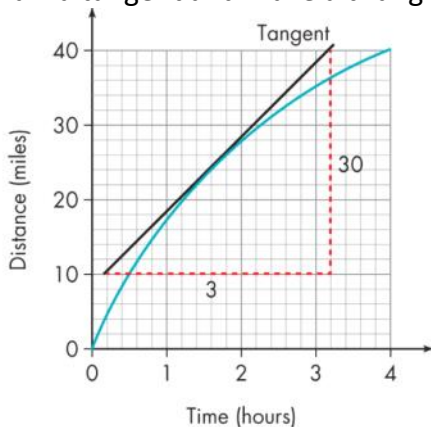
Gradient

- Gradient (m) = $\frac{\text{Difference in } y (\Delta y)}{\text{Difference in } x (\Delta x)}$
- Gradient measures the *rate of change* in y over x
- For a straight line, the gradient is a _____.
E.g. the gradient of $y = 5x + 6$ is always ____.
- For a curve, the gradient _____ - as different part of a curve has different steepness!

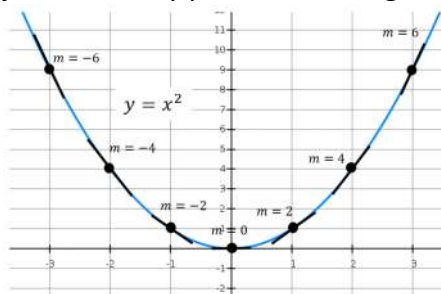
So how do we find the gradient of a particular point on a curve?

GCSE way:

- Draw a tangent and make a triangle to *estimate* the gradient on any point



- But we know we cannot rely on this method to work out gradient on a curve - there are just too many points and the gradient is ever-changing!



A-level way:

- Find the _____ through differentiation. It is a function that can be used to find the *gradient* of the curve *at any given point*.



12.3 - Differentiating x^n

Notes

- The gradient function/derivative of $y = f(x)$ is denoted as
- We can perform *differentiation* to find the derivative:

$$\text{If } y = x^n, \text{ then } f'(x) \text{ or } \frac{dy}{dx} =$$

$$\text{If } y = ax^n, \text{ then } f'(x) \text{ or } \frac{dy}{dx} =$$

Example

Find the derivatives $\left(\frac{dy}{dx}\right)$ of the following functions:

a) $y = x^5$

b) $f(x) = x^{\frac{1}{2}}$

c) $y = 2x^6$

d) $y = x^2 \times x^3$

e) $f(x) = \frac{x^3}{x^7}$

f) $y = \sqrt{25x^5}$

Practice

Find $\frac{dy}{dx}$ when y equals:

a) $7x^3$

b) $3x^{-10}$

c) $-3x^{\frac{3}{2}}$

d) $\frac{x^3}{x^{\frac{1}{2}}}$

e) $\sqrt{81x^4}$

f) $\frac{10x^3 \times 2x^9}{5x}$

12.4 - Differentiating quadratics

Notes

- When a function has more than one term, we can differentiate the terms *one-at-a-time*.

Example

Differentiate $y = 2x^2 + 5x - 6$

Practice Q1

Find $\frac{dy}{dx}$ for the following functions:

- $y = x^2 + 4x$
- $y = 2x^2 - 3x$
- $y = 2 - 5x^4$
- $y = 2x + px^2 - 3$, where p is a constant.

Example

Let $f(x) = 4x^2 - 8x + 3$

- Find the gradient of $y = f(x)$ at the point $(\frac{1}{2}, 0)$
- Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 24.
- Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.



12.4 - Differentiating quadratics

Practice Q2

Let $f(x) = x^2 - 4x + 2$

- Find the gradient of $y = f(x)$ at the point $(1, -1)$
- Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 5.
- Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 2 - x$.

Practice Q3

Find the gradient of the curve with equation $y = 2x^2 - 3x + 1$ at the point $(2, 3)$.

Practice Q4

Find the gradients of the curve $y = 3x^2$ at the points C and D where the curve meets the line $y = x + 4$.



12.5 - Differentiating harder functions

Notes

You will be required to differentiate "harder" functions, they usually include:

- Roots (Turn into power)
e.g. $y = \sqrt[3]{x}$
- Fractions (Split them up)
e.g. $y = \frac{x^2+3}{\sqrt{x}}$
- Brackets (expand them)
e.g. $y = x^2(x - 3)$
- Numbers in denominator
e.g. $y = \frac{1}{3x}$

Example (Roots)

Find $\frac{dy}{dx}$ when $y = \sqrt[3]{x}$

Example (Fractions)

Find $\frac{dy}{dx}$ when $y = \frac{x^2+3}{\sqrt{x}}$

Example (Brackets)

Find $\frac{dy}{dx}$ when $y = x^2(x - 3)$

Example (Numbers in denominator)

Find $\frac{dy}{dx}$ when $y = \frac{1}{3x}$



12.5 - Differentiating harder functions

Practice Q1

Find $\frac{dy}{dx}$ for the following functions of y

a) $y = \frac{1}{\sqrt{x}}$

b) $y = x^2(2x^4 + 5)$

c) $y = \frac{3 + x^4}{x^2}$

d) $y = \frac{1 - 2x}{3\sqrt{x}}$

Practice Q2

Find the gradient of $y = 3x - \frac{4}{x^3}$ at the point (2, 5.5)

Practice Q3

$f(x) = \frac{12}{p\sqrt{x}} + x$, where p is a real constant.

Given that $f'(2) = 3$, find p and give your answer in the form $a\sqrt{2}$.



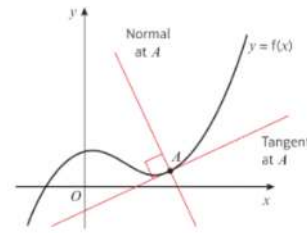
12.6 - Gradients, tangents and normals

Notes

- The *tangent* to a curve is a straight line that "just touches" the curve.
- The *normal* is a straight line that is perpendicular to the tangent.

Finding gradient of tangent of a particular point

1. Differentiate to find the gradient function
- 2.



Example

Find the gradient of the tangent to the curve $y = x^3 - 3x^2 + 2x - 1$ at the point (3,5).

Finding the equation of tangent

1. Find the gradient of tangent (use steps above)
2. Work out equation of straight line as usual ($y - y_1 = m(x - x_1)$ or $y = mx + c$).

Example

Find the equation of tangent to the curve $y = x^2 + 3x + 5$ when $x = 4$.

Practice Q1

Find the equation of tangent to the curve $y = x + 3\sqrt{x}$ when $x = 9$.



12.6 - Gradients, tangents and normals

Example

Find the equation of **normal** to the curve with equation $y = 4 + 3x^3$ when $x = -2$.

Practice Q2

Find the equation of **normal** to the curve $y = x + \frac{1}{x}$ when $x = 2$.

Practice Q3

The curve C has the equation $y = x^3 - x^2 + 2x - 4$.

- Find the equation of tangent to C when $x = 1.5$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.
- Find the equation of normal to C at the point $(2, 4)$. Give your answer in the form $y = ax + b$.



12.7 - Increasing/Decreasing functions

Notes

We can use the derivative to determine if a function is **increasing** or **decreasing** in a given interval.

- The function is *increasing* if
- The function is *decreasing* if



Example

Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

Practice Q1

Find the interval on which the function $f(x) = x^3 + 6x^2 - 135x$ is increasing and write your answer in set notation.

Example

Show that the function $f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x .



12.7 - Increasing/Decreasing functions

Example

Show that the function $f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x .

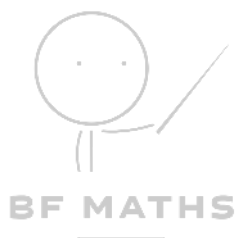
Practice Q2

Show that the function $f(x) = x^3 + 15x - 2$ is increasing for all real values of x .

Exam practice

$$f(x) = x^3 - 4x^2 - 35x + 20$$

Find the set of values of x for which $f(x)$ is increasing. **(4 marks)**

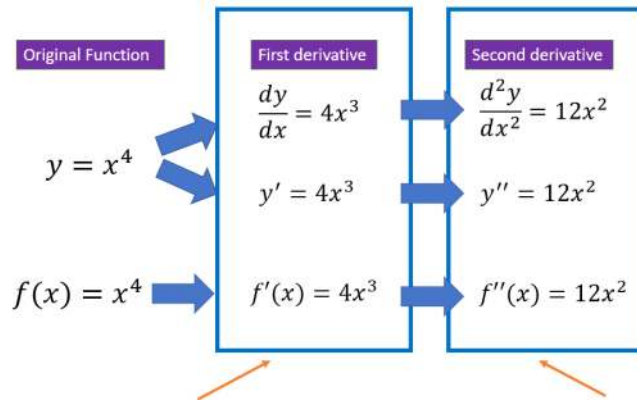


12.8 - Second order derivatives

Notes

When you differentiate the function once, it is known as first derivative.

When differentiating the second time, it is known as second derivative.



Example

Given that $y = 3x^5 + \frac{4}{x^2}$, find

- $\frac{dy}{dx}$
- $\frac{d^2y}{dx^2}$

Practice Q1

If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.

Practice Q2

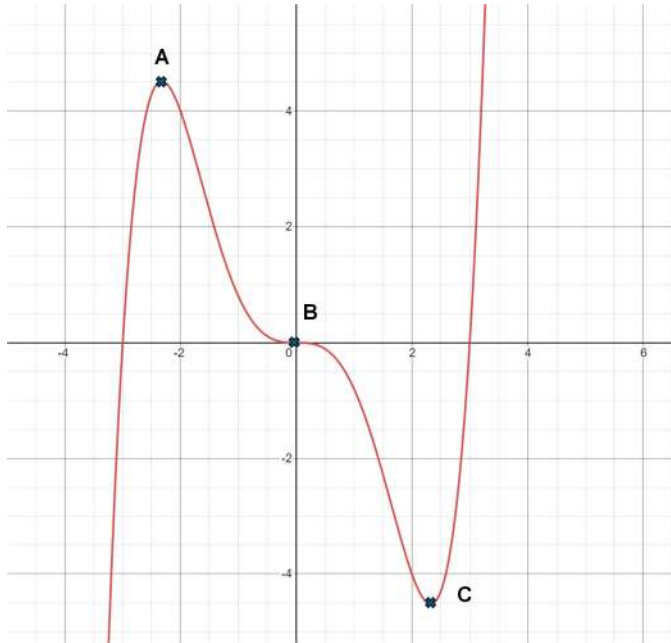
If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.



12.9 - Stationary points

Notes

- A *stationary point* on a curve is any point where the curve has _____.
- For any stationary point:
- We can determine whether a stationary point is a



- What are the stationary points?

**It is local because it is max/min within the vicinity.
There is also global max/min in some graphs.

Ways to determine the nature of stationary point

1. Look for the change in signs of the gradients before and after the SP.
2. Use second derivative (preferrable)



12.9 - Stationary points

Example (Method 1)

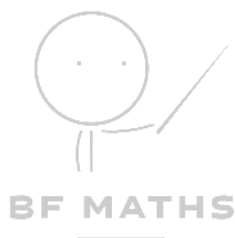
Given one of the stationary points on the curve with equation $y = x^4 - 32x$ is $(2, -48)$. Determine whether it is a local maximum, a local minimum or a point of inflection.

Example (Method 2)

- Find the coordinates of the stationary point on the curve with equation $y = 2x^3 - 15x^2 + 24x + 6$.
- Determine the nature of each stationary point.

Practice Q1

Find the turning point of the curve $y = x - 3\sqrt{x}$ and determine the nature of the turning point.



12.9 - Stationary points

Practice Q2

Given $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$,

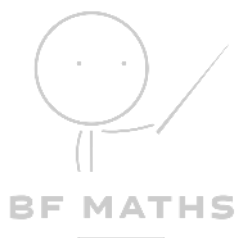
- Find the coordinates of all turning points and determine the nature of each.
- Sketch the graph of $y = f(x)$.

Exam Practice

The curve with equation: $y = x^2 - 32\sqrt{x} + 20$, $x > 0$, has a stationary point P .

Use calculus

- to find the coordinates of P . **(6 marks)**
- to determine the nature of the stationary point P . **(3 marks)**



12.10 - Sketching gradient functions

The new A-level specifically mentions the skill to sketch the gradient function of a curve (i.e. $y = f'(x)$).

If you know the original curve ($y = f(x)$), then you can differentiate to find the gradient function $y = f'(x)$ and sketch the graph using the equation easily.

However, in most cases, you will not be given the equation of original curve.

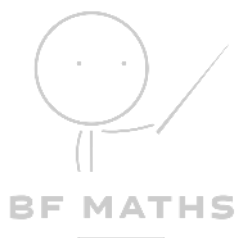
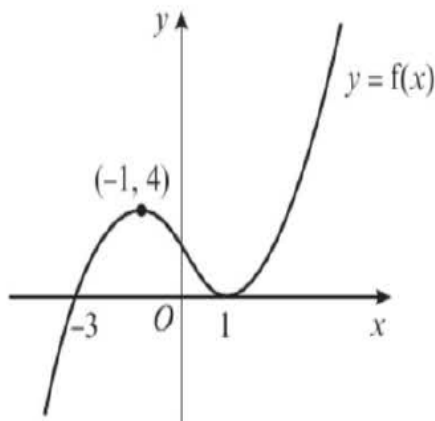
Notes

- Draw a lot of mini tangent lines to help observe the change in gradient

<u>Original graph</u>	<u>Gradient function graph</u>
x-axis: x y-axis: $f(x)$	x-axis: x y-axis: $f'(x)$
Positive gradient	Above the x-axis
Stationary point	x-intercepts
Negative gradient	Below the x-axis

Example

The diagram shows the curve with equation $y = f(x)$. The curve has stationary points at $(-1, 4)$ and $(1, 0)$, and cuts the x-axis at $(-3, 0)$. Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x-axis.

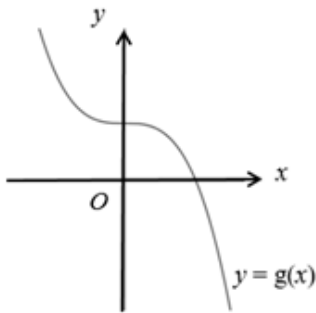


12.10 - Sketching gradient functions

Example

The diagram shows the graph of $y = g(x)$.

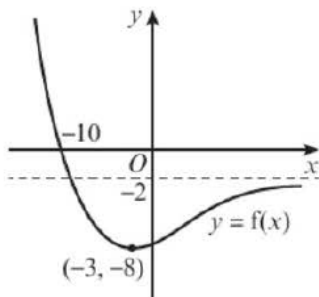
Sketch the graph of the gradient function for the graph of $y = g(x)$.



Example / Practice Q1

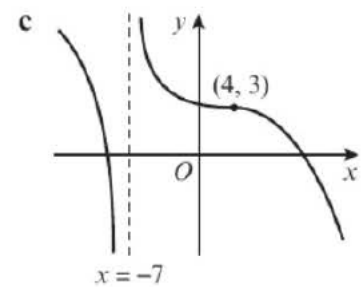
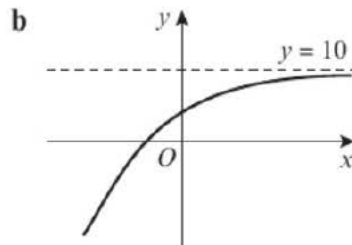
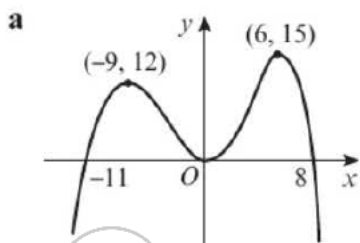
The diagram shows the curve with equation $y = f(x)$. The curve has an asymptote at $y = -2$ and a turning point at $(-3, -8)$. It cuts the x -axis at $(-10, 0)$.

- Sketch the graph of $y = f'(x)$.
- State the equation of asymptote of $y = f'(x)$.



Practice Q2

For each graph given, sketch the graph of the corresponding gradient function. Show the coordinates of any points where the curve cuts or meets the x -axis and give the equation of any asymptotes.



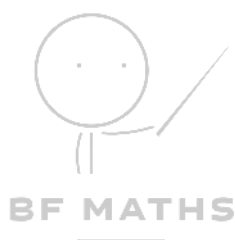
12.11 - Modelling with differentiation

Notes

-
-
-
- In differentiation, modelling questions usually come in 2 types:
 - rate of change
 - optimisation (finding max/min. of an object)

Example (Rate of change)

Given that the volume, $V \text{ cm}^3$, of an expanding sphere is related to its radius, $r \text{ cm}$, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to the radius at the instant when the radius is 5cm.



12.11 - Modelling with differentiation

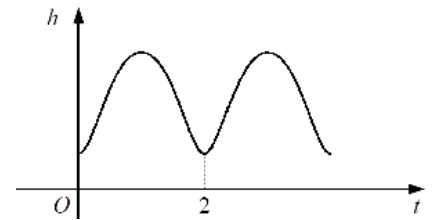
Practice Q1

The graph shows the height, h cm, of the letters on a website advert t seconds after the advert appears on the screen.

For t in the interval, $0 \leq t \leq 2$, h is given by the equation: $h = 2t^4 - 8t^3 + 8t^2 + 1$.

For larger values of t , the variation of h over this interval is repeated every 2 seconds.

- Find $\frac{dh}{dt}$ for t in the interval $0 \leq t \leq 2$.
- Find the rate at which the height of the letters is increasing when $t = 0.25$
- Find the maximum height of the letters.



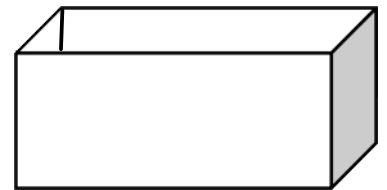
Example (Optimisation)

A large tank in the shape of a cuboid is to be made from 54m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

- Show that the volume, $V \text{ m}^3$, of the tank is given by

$$V = 18x - \frac{2}{3}x^3$$

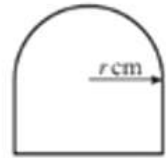
- Given that x can vary, use differentiation to find the maximum or minimum of V .
- Determine the value of V you have found is a maximum or minimum.



12.11 - Modelling with differentiation

Example (Optimisation)

The Among Us character consists of a rectangular base with a semi-circular top, as shown.



- a) Given that the perimeter of the shape is 40cm, show that its area,

$$A \text{ cm}^2, \text{ is given by the formula } A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

(4 marks)

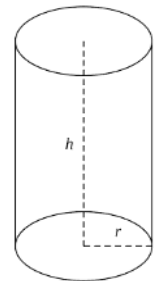
- b) Hence find the maximum value for the area of the shape.

(4 marks)

Practice Q2

A cylindrical can with height h metres and radius r metres has a capacity of 1000cm^3 .

- a) Find an expression for the surface area of the can in terms of r only.
b) Find the value of r which minimises the surface area of the can and hence find the minimum surface area.

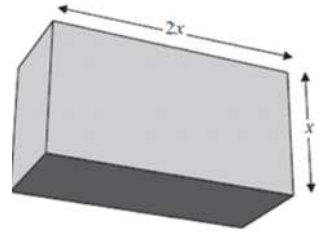


12.11 - Modelling with differentiation

Exam Practice

A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm, as shown the figure.

The volume of the cuboid is 81 cm^3 .

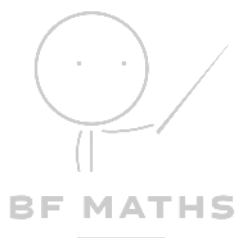


a) Show that the total length, L cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2} \quad \text{(3 marks)}$$

b) Use calculus to find the minimum value of L . **(6 marks)**

c) Justify, by further differentiation, that the value of L that you have found is a minimum. **(2 marks)**



12.2 - Differentiation by first principles

Starter (Substitution)

Work out the following y -value in each function:

- 1) $f(x) = x^2$, $x = 5$
- 2) $f(x) = 2x + 1$, $x = -3$
- 3) $f(x) = x^3 - 4$, $x = 2$
- 4) $f(x) = 2x^2$, $x = (x + h)$

Background

- As we know differentiation helps us find the gradient function (and hence calculate the gradient) about a certain point on the curve, there is also a method called differentiation by first principles derived by two very famous mathematicians *Issac Newton* and *Gottfried Leibniz*.
- The shortcut we learnt ($f'(x) = nx^{n-1}$) is also derived from first principles.

Notes

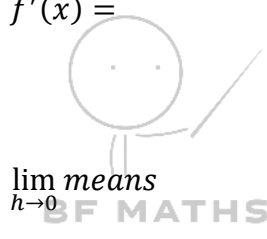
- The main premise is that say we want to find the gradient function of a point on a curve $y = f(x)$, let the point to be $(x, f(x))$. If we draw the tangent line *extremely small* (imagine we zoom in a curve using a microscope), then that part of the curve will become almost like a straight line, just like the tangent. We can then find the gradient like we do for straight line - $\frac{\text{difference in } y}{\text{difference in } x}$



Differentiation by first principles formula

Given $y = f(x)$, then

$$f'(x) =$$



12.2 - Differentiation by first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example

Differentiate $y = x^2$ using first principles.

Example

Find $f'(x)$ given $f(x) = 4x^2 - 3x + 5$ using first principles.

Practice Q1

Prove, using first principles, the derivative of $f(x) = 4x^2$ is $8x$.



12.2 - Differentiation by first principles

Practice Q2

Prove, using first principles, the derivative of $f(x) = x^2 + 3x$ is $2x + 3$.

Practice Q3**

Differentiate $f(x) = x - \frac{1}{x}$ using first principles.

