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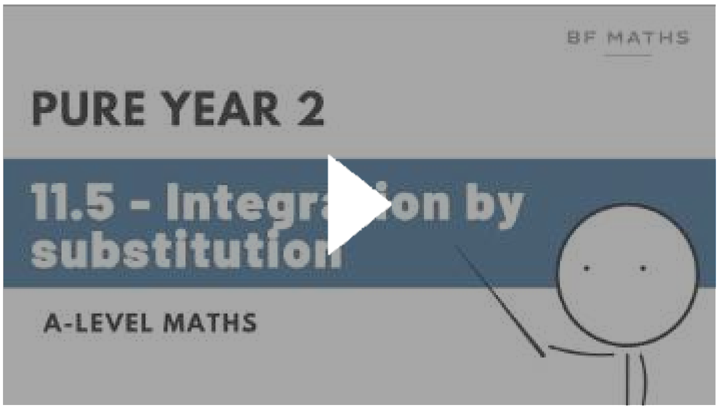
If you need help on this chapter:
A-Level Maths | Pure Year 2 | 11.5 - Integration by substitution Walkthrough | Edexcel



11.5 - Integration by substitution

1a) $u = x+2$

$\frac{du}{dx} = 1$

1b) $u = x+2$

$x = u-2$

$2x = 2(u-2) = \underline{\underline{2u-4}}$

1c) $\int 2x(x+2)^4\,dx$ $\qquad \frac{du}{dx} = 1$, $du = dx$

$= \int (2u-4)(u)^4\,du$

$= \int 2(u-2)u^4\,du$

1d) $\int 2(u-2)u^4\,du$

$= 2\int (u-2)(u)^4\,du$

$= 2\int u^5 - 2u^4\,du$

$= 2\left[\frac{u^6}{6} - \frac{2u^5}{5}\right] + c$

$= \frac{u^6}{3} - \frac{4u^5}{5} + c$

1e) Sub $u = x+2$

$$= \frac{(x+2)^6}{3} - \frac{4(x+2)^5}{5} + c$$

2a) $\int x(x-8)^4\,dx$; $u = x-8$

$x = u+8$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\int x(x-8)^4\,dx = \int (u+8)(u)^4\,du$

$= \int u^5 + 8u^4\,du$

$= \frac{u^6}{6} + \frac{8u^5}{5} + c$

$$= \frac{(x-8)^6}{6} + \frac{8(x-8)^5}{5} + c$$

2b) $\int \sin x\cos x(1+\cos x)^5\,dx$; $u = 1+\cos x$

$\cos x = u-1$

$\frac{du}{dx} = -\sin x \Rightarrow dx = \frac{du}{-\sin x}$

$\int \text{''}\,dx = \int \cancel{\sin x}(u-1)(u)^5\,\frac{du}{-\cancel{\sin x}}$

$= \int -(u-1)u^5\,du$

$= \int -u^6 + u^5\,du$

$= -\frac{u^7}{7} + \frac{u^6}{6} + c$

$$= \frac{-(1+\cos x)^7}{7} + \frac{(1+\cos x)^6}{6} + c$$

11.5 - Integration by substitution

2c) $\int \operatorname{cosec}^2 x \cot x \sqrt{2+\cot x}\, dx$; $u = 2+\cot x$

$\rightarrow u-2 = \cot x$

$\cot x = u-2$

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$\frac{du}{-\operatorname{cosec}^2 x} = dx$

$= \int \cancel{\operatorname{cosec}^2 x}(u-2)\sqrt{u} \times \frac{du}{\cancel{-\operatorname{cosec}^2 x}}$

$= \int -(u-2)u^{\frac{1}{2}}\, du$

$= \int -u^{\frac{3}{2}} + 2u^{\frac{1}{2}}$

$= -\frac{2}{5}u^{\frac{5}{2}} + 2\times\frac{2}{3}u^{\frac{3}{2}}$

$= \underline{\underline{-\frac{2}{5}(2+\cot x)^{\frac{5}{2}} + \frac{4}{3}(2+\cot x)^{\frac{3}{2}} + c}}$

2d) $\int \frac{x}{\sqrt{x^2+1}}\, dx$; $u = x^2+1 \rightarrow \frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$dx = \frac{du}{2x}$

$= \int \frac{\cancel{x}}{\sqrt{u}} \times \frac{du}{2\cancel{x}}$

$= \int \frac{1}{2u^{\frac{1}{2}}}\, du \quad = \int \frac{1}{2}u^{-\frac{1}{2}}\, du$

$= \frac{1}{2}\times 2u^{\frac{1}{2}} + c$

$= u^{\frac{1}{2}} + c$

$= \underline{\underline{(x^2+1)^{\frac{1}{2}} + c}}$

11.5 - Integration by substitution

3a) $\int_{\frac{1}{2}}^{\frac{5}{4}} x\sqrt{4x-1}\,dx$; $u = 4x-1$ $\rightarrow$ $\frac{du}{dx} = 4$, $\frac{du}{4} = dx$

$u+1 = 4x$

$x = \frac{u+1}{4}$

x	$\frac{1}{2}$	$\frac{5}{4}$
u	$4\times\frac{1}{2}-1$ $=1$	$4\times\frac{5}{4}-1$ $=4$

$$= \int_1^4 \frac{(u+1)}{4} \times \sqrt{u} \times \frac{du}{4}$$

$$= \int_1^4 \frac{u^{\frac{3}{2}} + u^{\frac{1}{2}}}{16}\,du$$

$$= \frac{1}{16}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_1^4 = \frac{1}{16}\left[\frac{272}{15} - \frac{16}{15}\right] = \underline{\underline{\frac{16}{15}}}$$

3b) $\int_{16}^{25} \frac{4}{\sqrt{x}(\sqrt{x}-9)}\,dx$; $u = \sqrt{x}$ $\rightarrow$ $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$

$\frac{du}{dx} = \frac{1}{2u}$

$2u\,du = dx$

$dx = 2u\,du$

x	16	25
u	$\sqrt{16}$ $=4$	$\sqrt{25}$ $=5$

$$= \int_4^5 \frac{4}{\cancel{u}(u-9)} \times 2\cancel{u}\,du$$

$$= \int_4^5 \frac{8}{u-9}\,du$$

$$= 8\int_4^5 \frac{1}{u-9}\,du$$

RcR $\rightarrow$ Let $y = \ln|u-9|$

$\frac{dy}{du} = \frac{1}{u-9}$

$$= 8\left[\ln|u-9|\right]_4^5$$

$$= 8\left[\ln|5-9| - \ln|4-9|\right]$$

$$= 8\left[\ln 4 - \ln 5\right] = \underline{\underline{8\ln\frac{4}{5}}}$$

3c) $\displaystyle\int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$; $u = \cos x \rightarrow \frac{du}{dx} = -\sin x$

$\frac{du}{-\sin x} = dx$

$dx = \frac{du}{-\sin x}$

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
u	$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$	$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$= \int_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}} \frac{\cancel{\sin x}}{u^3} \times \frac{du}{-\cancel{\sin x}}$$

$$= \int -u^{-3}\,du = \left[\frac{u^{-2}}{2}\right]_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}} = \left[\frac{\left(-\frac{\sqrt{2}}{2}\right)^{-2}}{2} - \frac{\left(-\frac{1}{2}\right)^{-2}}{2}\right]$$

$$= 1 - 2$$

$$= \underline{\underline{-1}}$$

3d) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin 2x}{1+\sin x}\,dx$; $u = \sin x \rightarrow \frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

x	0	$\frac{\pi}{2}$
u	$\sin(0) = 0$	$\sin\left(\frac{\pi}{2}\right) = 1$

$$= \int_0^1 \frac{2\sin x\cancel{\cos x}}{1+u}\,\frac{du}{\cancel{\cos x}}$$

$$= \int \frac{2u}{1+u}\,du = 2\int \frac{u}{1+u}\,du$$

Split into partial fraction

$$\begin{array}{r} 1 \\ u+1 \,\big)\overline{\;u\;\;\;\;\;\;} \\ -)\,u+1 \\ \hline -1 \end{array}$$

$$= 2\int 1 - \frac{1}{u+1}\,du$$

$$= 2\left[u - \ln|u+1|\right]_0^1$$

$$= 2\left[1 - \ln 2 - (0 - \ln 1)\right]$$

$$= \underline{\underline{2 - 2\ln 2}} \quad \text{or} \quad 2 - \ln 4$$

$2\ln 2 = \ln(2^2)$

4) $\int_0^2 x^3\sqrt{1+x^2}\,dx$; $u=1+x^2 \rightarrow \frac{du}{dx}=2x$

$u-1=x^2$

$x^2=u-1$

$\frac{du}{2x}=dx$

$dx=\frac{du}{2x}$

x	0	2
u	$1+0^2=1$	$1+2^2=5$

$$=\int_1^5 x^{\overset{2}{\cancel{3}}}(u)^{\frac{1}{2}}\times\frac{du}{2\cancel{x}}$$

$$=\frac{1}{2}\int_1^5 x^2(u)^{\frac{1}{2}}\,du$$

$$=\frac{1}{2}\int_1^5 (u-1)(u)^{\frac{1}{2}}\,du$$

$$=\frac{1}{2}\int_1^5 u^{\frac{3}{2}}-u^{\frac{1}{2}}\,du$$

$$=\frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]_1^5=\frac{1}{2}\left[\left(\frac{2}{5}(5)^{\frac{5}{2}}-\frac{2}{3}(5)^{\frac{3}{2}}\right)-\left(\frac{2}{5}(1)^{\frac{5}{2}}-\frac{2}{3}(1)^{\frac{3}{2}}\right)\right]$$

Calculator can't compute the exact value.
do $\frac{2}{5}(5)^2\times\sqrt{5}$ instead

$$=\frac{1}{2}\left(10\sqrt{5}-\frac{10}{3}\sqrt{5}-\left(\frac{2}{5}-\frac{2}{3}\right)\right)$$

$$=\frac{2+50\sqrt{5}}{15} \quad \text{or} \quad \frac{2}{15}+\frac{10\sqrt{5}}{3}$$

11.5 - Integration by substitution

5) $\displaystyle\int_{\ln 2}^{\ln 6} \frac{1}{1+e^x}\,dx$; $u = e^x$ $\rightarrow \dfrac{du}{dx} = e^x$

$dx = \dfrac{du}{e^x}$

$dx = \dfrac{du}{u}$

x	$\ln 2$	$\ln 6$
u	$=e^{\ln 2}$ $=2$	$=e^{\ln 6}$ $=6$

$$= \int_2^6 \frac{1}{1+u} \times \frac{du}{u}$$

$$= \int_2^6 \frac{1}{u(1+u)}\,du \xrightarrow[\text{partial fractions}]{\text{Split into}} \frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{u+1}$$

$1 = A(u+1) + Bu$

$(u=0)$: $1 = A$

$(u=-1)$: $1 = -B \Rightarrow B = -1$

$$\frac{1}{u(u+1)} \equiv \frac{1}{u} - \frac{1}{u+1}$$

$$= \int_2^6 \frac{1}{u} - \frac{1}{u+1}\,du$$

$$= \Big[\ln u - \ln(u+1)\Big]_2^6$$

$$= (\ln 6 - \ln 7) - (\ln 2 - \ln 3)$$

$$= \left(\ln \tfrac{6}{7}\right) - \left(\ln \tfrac{2}{3}\right)$$

$$= \ln\left(\tfrac{6}{7} \div \tfrac{2}{3}\right)$$

$$= \underline{\underline{\ln \tfrac{9}{7}}}$$

6a) Show that $\int \frac{1}{1+x^2}dx = \arctan x + c$

When $x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$ recall $1+\tan^2 A \equiv \sec^2 A$

$$\frac{dx}{d\theta} = 1+\tan^2\theta$$

$$\frac{dx}{d\theta} = 1+x^2$$

$$dx = (1+x^2)d\theta$$

$$\int \frac{1}{1+x^2}dx = \int \frac{1}{\cancel{1+x^2}} \times \cancel{(1+x^2)}d\theta$$

$$= \int 1\, d\theta$$

$$= \theta + c$$

$$= \underline{\underline{\arctan x + c}}$$

since $x = \tan\theta$

$\tan^{-1}x = \theta$

$\theta = \arctan x$

6b) $\int_{\frac{\sqrt{3}}{3}}^{1} \frac{1}{1+x^2}dx$

$$= \left[\arctan x\right]_{\frac{\sqrt{3}}{3}}^{1}$$

$$= \arctan(1) - \arctan\left(\frac{\sqrt{3}}{3}\right)$$

in radians because $y = \arctan x$ is defined in radians

recall graph of $\arctan x$

$$= \underline{\underline{\frac{\pi}{12}}}$$

11.5 - Integration by substitution

7) $\int_0^4 \frac{5x}{\sqrt{2x+1}}\,dx$; $u^2 = 2x+1$

$u = (2x+1)^{\frac{1}{2}} \rightarrow \frac{du}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2$

$\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{(2x+1)^{\frac{1}{2}}}$

$dx = (2x+1)^{\frac{1}{2}}\,du$

$dx = u\,du$

x	0	4
u	1	3

$= \int_1^3 \frac{5x}{\cancel{\sqrt{u^2}}} \times \cancel{u}\,du$

$= \int_1^3 5x\,du$

$u^2 = 2x+1$

$u^2 - 1 = 2x$

$x = \frac{u^2-1}{2}$

$= \int_1^3 5\left(\frac{u^2-1}{2}\right)du$

$= \frac{5}{2}\int_1^3 u^2 - 1\,du$

$= \frac{5}{2}\left[\frac{u^3}{3} - u\right]_1^3$

$= \frac{5}{2}\left[\left(\frac{(3)^3}{3} - 3\right) - \left(\frac{1}{3} - 1\right)\right]$

$= \frac{5}{2}\left(6 - -\frac{2}{3}\right)$

$= \underline{\underline{\frac{50}{3}}}$

11.5 - Integration by substitution

8) $\int \frac{2\sin 2x}{1+\sin x}\,dx$

Let $u = 1+\sin x \rightarrow \frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$= \int \frac{2(2\sin x \cancel{\cos x})}{u} \times \frac{du}{\cancel{\cos x}}$$

$$= \int \frac{4\sin x}{u}\,du$$

$$= 4\int \frac{u-1}{u}\,du$$

$$= 4\int \frac{u}{u} - \frac{1}{u}\,du$$

$$= 4\int 1 - \frac{1}{u}\,du$$

$$= 4\left[u - \ln u\right] + c$$

$$= 4u - 4\ln u + c$$

$$= \underline{\underline{4(1+\sin x)^{*} - 4\ln|1+\sin x| + c}}$$

* The book answer is $4\sin x$ because if you expand the bracket $4(1+\sin x)$, you get $4+4\sin x$, which "4" is a constant, which can be ignored because there is a "+c" to represent all constants.

So both answers are correct.