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9.9 Using Second derivatives

1a] $f(x) = e^{-x}\cos 2x$

$u = e^{-x}$ $\quad$ $v = \cos 2x$

$\frac{du}{dx} = -e^{-x}$ $\quad$ $\frac{dv}{dx} = 2(-\sin 2x)$

$\frac{dv}{dx} = -2\sin 2x$

$\frac{du}{dx} = -2e^{-x}\sin 2x - e^{-x}\cos 2x$

$f'(x) = -2e^{-x}\sin 2x - e^{-x}\cos 2x$

b] $f'(x) = -2e^{-x}\sin 2x - e^{-x}\cos 2x$

$-2e^{-x}\sin 2x$

$u = -2e^{-x}$ $\quad$ $v = \sin 2x$

$\frac{du}{dx} = 2e^{-x}$ $\quad$ $\frac{dv}{dx} = 2(\cos 2x)$

$\frac{dv}{dx} = 2\cos 2x$

$\frac{du}{dx} = 2e^{-x}\sin 2x - 4e^{-x}\cos 2x$

$-e^{-x}\cos 2x$

$u = -e^{-x}$ $\quad$ $v = \cos 2x$

$\frac{du}{dx} + e^{-x}$ $\quad$ $\frac{dv}{dx} = 2(-\sin 2x)$

$\frac{dv}{dx} = -2\sin 2x$

$\frac{du}{dv} = e^{-x}\cos 2x + 2e^{-x}\sin 2x$

9.9 Using Second derivative

1b] Cont.

$f''(x) = 2e^{-x}\sin 2x - 4e^{-x}\cos 2x + e^{-x}\cos 2x + 2e^{-x}\sin 2x$

$f''(x) = 4e^{-x}\sin 2x - 3e^{-x}\cos 2x$

$f''(x) = e^{-x}(4\sin 2x - 3\cos 2x)$

2a] $f(x) = 8x^2 + 3e^{4x}$

$f'(x) = 16x + 12e^{4x}$

$f''(x) = 16 + 48e^{4x}$

b] $16 \geq 0$ and $48e^{4x} \geq 0$ for all values of x, hence $f(x)$ is convex for all values of x.

3a] $f(x) = \cos 2x$

$f'(x) = 2(-\sin 2x)$

$f'(x) = -2\sin 2x$

$f''(x) = -4(\cos 2x)$

$f''(x) = -4\cos 2x$

b] $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

4a] $f(x) = x(2x-1)^5$

9.9 Using Second derivatives

4a] Cont.

$u = x \qquad\qquad v = (2x-1)^5$

$\frac{du}{dx} = 1 \qquad\qquad \frac{dv}{dx} = 5(2x-1)^4 \times 2$

$\frac{dv}{dx} = 10(2x-1)^4$

$\frac{du}{dv} = 10x(2x-1)^4 + (2x-1)^5$

$f'(x) = 10x(2x-1)^4 + (2x-1)^5$

$u = 10x \qquad\qquad v = (2x-1)^4$

$\frac{du}{dx} = 10 \qquad\qquad \frac{dv}{dx} = 4(2x-1)^3 \times 2$

$\frac{dv}{dx} = 8(2x-1)^3$

$\frac{du}{dx} = 80x(2x-1)^3 + 10(2x-1)^4$

$$\begin{aligned} f''(x) &= 80x(2x-1)^3 + 10(2x-1)^4 + 5(2x-1)^4 \times 2 \\ &= 80x(2x-1)^3 + 10(2x-1)^4 + 10(2x-1)^4 \\ &= 80x(2x-1)^3 + 20(2x-1)^4 \\ &= 20\left(4x(2x-1)^3 + (2x-1)^4\right) \\ &= 20(2x-1)^3(4x + 2x - 1) \\ &= 20(2x-1)^3(6x-1) \end{aligned}$$

b] $f''(x) = 20(2x-1)^3(6x-1)$

$f''\left(\frac{1}{2}\right) = 20\left(2\left(\frac{1}{2}\right)-1\right)^3\left(6\left(\frac{1}{2}\right)-1\right)$

$= 0$

9.9 Using Seccond derivative

4b] Cont.

$f''\left(\frac{1}{6}\right) = 20\left(2\left(\frac{1}{6}\right)-1\right)^3\left(6\left(\frac{1}{6}\right)-1\right)$

$= 0$

Hence, proved.

c] $f''(0.1) = 20(2(0.1)-1)^3(6(0.1)-1)$
$= 4.096 > 0$

$f''(0.2) = 20(2(0.2)-1)^3(6(0.2)-1)$
$= -0.864 < 0$

As there is a sign change, this confirm there is a point of inflection at $x = \frac{1}{6}$

5] $f(x) = x^3 - 8x^2 + 6x - 2$
$f'(x) = 3x^2 - 16x + 6$
$f''(x) = 6x - 16$

$6x - 16 \geqslant 0$
$6x \geqslant 16$
$x \geqslant \frac{16}{6}$

$x \geqslant \frac{8}{3}$

6] $f(x) = x(2x-5)^4$

9.9 Using Second derivative

6] Cont.

a) $u = x$ $\qquad$ $v = (2x-5)^4$

$\frac{du}{dx} = 1$ $\qquad$ $\frac{dv}{dx} = 4(2x-5)^3 \times 2$

$\frac{dv}{dx} = 8(2x-5)^3$

$\frac{du}{dv} = 8x(2x-5)^3 + (2x-5)^4$

$f'(x) = 8x(2x-5)^3 + (2x-5)^4$

$u = 8x$ $\qquad$ $v = (2x-5)^3$

$\frac{du}{dx} = 8$ $\qquad$ $\frac{dv}{dx} = 3(2x-5)^2 \times 2$

$\frac{dv}{dx} = 6(2x-5)^2$

$\frac{du}{dv} = 48x(2x-5)^2 + 8(2x-5)^3$

$f''(x) = 48x(2x-5)^2 + 8(2x-5)^3 + 8(2x-5)^3$

$f''(x) = 48x(2x-5)^2 + 16(2x-5)^3$

b) let $f''(x) = 0$

$48x(2x-5)^2 + 16(2x-5)^3 = 0$

$(2x-5)^2(48x + 16(2x-5)) = 0$

$(2x-5)^2(48x + 32x - 80) = 0$

$(2x-5)^2(80x - 80) = 0$

So either

$(2x-5)^2 = 0$

$2x - 5 = 0 \Longrightarrow x = \frac{5}{2}$

9.9 Using Second derivative

6b] Cont.

OR

$$~~4x~~ \quad 80x - 80 = 0$$

$$80x = 80$$

$$x = 1$$

Sub $x = 1$ in $f''(x)$

$$f''(1) = 48(1)((2(1)) - 5)^2 + 16(2(1) - (5))^3$$

$$= 0$$

Sub $x = \frac{5}{2}$ in $f''(x)$

$$f''(x) = 48\left(\frac{5}{2}\right)\left(2\left(\frac{5}{2}\right) - 5\right)^2 + 16\left(2\left(\frac{5}{2}\right) - 5\right)^3$$

$$f''(x) = 0$$

Since $f''(1) = 0$ and the second derivative changes sign around $x = 1$, there is a point of inflection ~~at~~ at $x = 1$

7] $y = f(x)$ $\qquad f(x) = \dfrac{\sin 4x}{e^{2x}}$

$u = \sin 4x$ $\qquad v = e^{2x}$

$\dfrac{du}{dx} = 4\cos 4x$ $\qquad \dfrac{dv}{dx} = 2e^{2x}$

$$\frac{du}{dx} = \frac{4e^{2x}\cos 4x - 2e^{2x}\sin 4x}{e^{4x}}$$

9.9 Using Second derivative

7] Cont.

$$\frac{du}{dv} = \frac{e^{2x}(4\cos 4x - 2\sin 4x)}{e^{4x}}$$

$$\frac{du}{dv} = \frac{4\cos 4x - 2\sin 4x}{e^{2x}}$$

$$f'(x) = \frac{4\cos 4x - 2\sin 4x}{e^{2x}}$$

$u = 4\cos 4x - 2\sin 4x$ $\qquad$ $v = e^{2x}$

$\frac{du}{dv} = 16(-\sin 4x) - 8(\cos 4x)$ $\qquad$ $\frac{dv}{dx} = 2e^{2x}$

$$\frac{du}{dv} = -16\sin 4x - 8\cos 4x$$

$$\frac{du}{dv} = \frac{e^{2x}(-16\sin 4x - 8\cos 4x) - 2e^{2x}(4\cos 4x - 2\sin 4x)}{(e^{2x})^2}$$

$$\frac{du}{dv} = \frac{-16e^{2x}\sin 4x - 8e^{2x}\cos 4x - 8e^{2x}\cos 4x + 4e^{2x}\sin 4x}{(e^{2x})^2}$$

$$\frac{du}{dv} = \frac{-12e^{2x}\sin 4x - 16e^{2x}\cos 4x}{(e^{2x})^2}$$

$$\frac{du}{dv} = \frac{-12\sin 4x - 16\cos 4x}{e^{2x}}$$

$$f''(x) = \frac{-12\sin 4x - 16\cos 4x}{e^{2x}}$$

for $f(x)$ to be convex we require $f''(x) > 0$

9.9 Using Second derivative

7] Cont.

$-12\sin 4x - 16\cos 4x > 0$

This can be rearranged as

$12\sin 4x + 16\cos 4x < 0$

We can express this as $R\sin(4x+\alpha)$

$R = \sqrt{12^2+16^2}$

$R = 20$

The angle α can be found using $\tan\alpha = \frac{16}{12}$

$\tan\alpha = \frac{16}{12}$

$\tan\alpha = \frac{4}{3}$

$\alpha = 0.93$

$20\sin(4x+0.93) < 0$

$0 \le x \le \frac{\pi}{4}$

$\frac{\pi - 0.93}{4} < x < \frac{2\pi - 0.93}{4}$

$0.5529 < x < 1.3378$

Considering the domain:

$0 \le x \le \frac{\pi}{4}$

$\approx 0.7854.$

9.9 Using Second derivative

7] Cont.

$\therefore$ The interval where $f(x)$ is convex is $0.55 < x < 0.785$.

8] $f(x) = x^3 e^x$

$u = x^3$ $\qquad$ $v = e^x$

$\frac{du}{dx} = 3x^2$ $\qquad$ $\frac{dv}{dx} = e^x$

$\frac{du}{dv} = 3e^x x^2 + x^3 e^x$

$f'(x) = 3e^x x^2 + x^3 e^x$

$3e^x x^2$

$u = 3x^2$ $\qquad$ $v = e^x$

$\frac{du}{dx} = 6x$ $\qquad$ $\frac{dv}{dx} = e^x$

$\frac{du}{dv} = 6e^x x + 3x^2 e^x$

$x^3 e^x$

$u = x^3$ $\qquad$ $v = e^x$

$\frac{du}{dx} = 3x^2$ $\qquad$ $\frac{dv}{dx} = e^x$

$\frac{du}{dv} = 3x^2 e^x + x^3 e^x$

$f''(x) = 6e^x x + 3x^2 e^x + 3x^2 e^x + x^3 e^x$

$= e^x (x^3 + 6x^2 + 6x)$

9.9 Using Second derivative

8] Cont.

For the ~~concave~~ curve to be concave

$f''(x) > 0$

$e^x(x^3 + 6x^2 + 6x) > 0$

Since $e^x > 0$ for all values of x, we have:

$x(x^2 + 6x + 6) > 0$

the roots are

$x = 0$, $x = -3-\sqrt{3}$, $x = -3+\sqrt{3}$

$\therefore x \le -3-\sqrt{3}$ or

$-3+\sqrt{3} \le x \le 0$