Author: <u>Blinzy Fernandes</u>

This step-by-step solution guide has been created by **Blinzy Fernandes** for educational purposes. While we have made every effort to ensure the accuracy of the information presented, it is possible that there may be errors or omissions. We encourage users to critically evaluate and verify the content. BF Maths and the author cannot be held responsible for any errors or inaccuracies in this guide.

If you find any mistakes or have any suggestions for improvements, please contact us at bfmathshello@gmail.com. Your feedback is invaluable in helping us maintain the quality and accuracy of our resources. Please specify *which exercise and which question* in the email.

Thank you for using BF Maths for your maths revision!

1.1 Proof by Contradiction

1a] There are a finite number of prime numbers.

b] If there exist a number n such that if n^2 is even, then n must be odd.

c] There exist p and q such that if pq is odd then both p and q is even.

2] Assume that there is a greatest even number n.
Then $n+2>n$ and is also even.
This contradicts the assumption, so there is no greatest even number.

3] Assume that there exists integers a and b such that $a^2-4b-7=0$

So $a^2 = 7+4b$.
Since $4b+7$ is odd, 'a^2' is odd and so 'a' is odd and can be written as $2n+1$ where n is an integer.

Then $a^2 = (2n+1)^2 = 4b+7$
$\implies 4n^2+4n+1 = 4b+7$

3] Cont.

Rearranging $4n^2 + 4n - 4b = 6$
$4(n^2 + n - b) = 6$

So $n^2 + n - b = \frac{3}{2}$,

which is not an integer, but as 'n' and 'b' are integers, $n^2 + n - b$ must also be an integer.

This contradiction implies that $a^2 - 4b - 7$ cannot be equal to 0.

4] Assume that there is a smallest positive rational number n.
Then $n > 0$ and $n = \frac{a}{b}$,

where a and b are integers and $b \neq 0$

Then $\frac{n}{2} = \frac{a}{2b}$ and since b is an integer, 2b is also an integer.
This means that $\frac{n}{2}$ is rational and

as $\frac{n}{2} < n$, this is a contradiction,

which implies there is no smallest positive rational number.

1.1 Proof by Contradiction

5] Assume that if n^3 is odd, then n is even.

Let $n = 2K$

$n^3 = (2K)^3$

$n^3 = 8K^3$, which is even.

This is a contradiction.

Thus if n^3 is odd, then n must also be odd.

6] Assume that there exist non-zero integers x and y such that $x^2 - y^2 = 1$

then $(x+y)(x-y) = 1$

The only factor of 1 is 1,

The possible pairs are $(1,1)$ and $(-1,-1)$

$x + y = 1$ and $x - y = 1$ —— ①

$x + y = -1$ | $x - y = -1$ —— ②

$\Longrightarrow$ ① gives $x = 1$ and $y = 0$ and

$\Longrightarrow$ ② gives $x = -1$ and $y = 0$

$y = 0$ is a solutions in both cases so there are no non-zero integer solutions to the equations $x^2 - y^2 = 1$

7] Assume that $\sqrt{5}$ is rational.

Let $\sqrt{5} = \frac{a}{b}$ where a and b are integers, and $b \neq 0$.

The highest common factors of a and b is 1.

1.1 Proof by Contradiction

7] Cont.

$$(\sqrt{5})^2 = \left(\frac{a}{b}\right)^2$$

$$5 = \frac{a^2}{b^2} \qquad \Rightarrow a^2 = 5b^2$$

So a^2 is divisible by 5 and it must contain a (repeated) factor of 5 and so 'a' is divisible by 5.
So $a = 5p$ where p is an integer

$$(5p)^2 = \cancel{(5b)^2}\ 5b^2$$
$$\Longrightarrow 25p^2 = 5b^2$$
$$5p^2 = b^2$$

So b^2 is divisible by 5 and so b is divisible by 5. As 'a' and 'b' are both divisible by 5, then the highest common factor of 'a' & 'b' is not 1. This contradiction implies that $\sqrt{5}$ is irrational.

8] Assume that p is rational, q is irrational, and that $p-q$ is rational. Then there exist integers a, b, c and d such that

$$p = \frac{a}{b} \quad \text{and} \quad p - q = \frac{c}{d}$$

$$b, d \neq 0$$

1.1 Proof by Contradiction

8] Cont.

$$\text{So } \frac{a}{b} - q = \frac{c}{d}$$

$$\implies q = \frac{a}{b} - \frac{c}{d}$$

$$q = \frac{ad - cb}{bd}$$

But $ad - bc$ is an integer ~~and~~ as a, b, c, and d are all integers and $bd \neq 0$, which means that q is rational.

This is a contradiction.

So the difference between any rational number and any ~~rational~~ irrational number is irrational.