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Problem Solving: Set A

Bronze

$p(x) = x^3 - 9x^2 + 2x + 48$

a) Divide algebraically via long division:

$$
\begin{array}{r|l}
 & x^2 - 5x - 18 \\
\hline
x - 4 & x^3 - 9x^2 + 2x + 48 \\
 & \underline{x^3 - 4x^2} \\
 & \quad -5x^2 + 2x \\
 & \quad \underline{-5x^2 + 20x} \\
 & \qquad\quad -18x + 48 \\
 & \qquad\quad \underline{-18x + 72} \\
 & \qquad\qquad\quad -24
\end{array}
$$

Remainder $= -24$

b) $p(x) = x^3 - 9x^2 + 2x + 48$

$= (x-4)(x^2 - 5x - 18) - 24$

where $a = -5$, $b = -18$, and $r = -24$

c) $(x+2)$ is a factor $\therefore$ divide by $(x+2)$

$$
\begin{array}{r|l}
 & x^2 - 11x + 24 \\
\hline
x + 2 & x^3 - 9x^2 + 2x + 48 \\
 & \underline{x^3 + 2x^2} \\
 & \quad -11x^2 + 2x \\
 & \quad \underline{-11x^2 - 22x} \\
 & \qquad\quad 24x + 48 \\
 & \qquad\quad \underline{24x + 48} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\therefore p(x) = (x+2)(x^2 - 11x + 24)$

$= (x+2)(x-3)(x-8)$

Bronze

d) $3 < n < 8$

$n^3 - 9n^2 + 2n + 48 < 0$

Proof by exhaustion:

$n = 4 \quad = p(4) = 4^3 - 9(4)^2 + 2(4) + 48 = -24 < 0$

$n = 5 \quad = p(5) = 5^3 - 9(5)^2 + 2(5) + 48 = -42 < 0$

$n = 6 \quad = p(6) = 6^3 - 9(6)^2 + 2(6) + 48 = -48 < 0$

$n = 7 \quad = p(7) = 7^3 - 9(7)^2 + 2(7) + 48 = -36 < 0$

So for $3 < n < 8$, $x^3 - 9x^2 + 2x + 48 < 0$

Silver

$p(x) = 2x^3 - 15x^2 + 32x + 20$

$(2x+1)$ is a factor $\therefore$ divide algebraically by $2x+1$

a)
$$
\begin{array}{r|l}
 & x^2 - 8x + 20 \\
\hline
2x+1 & 2x^3 - 15x^2 + 32x + 20 \\
 & \underline{2x^3 + x^2} \\
 & \quad -16x^2 + 32x \\
 & \quad \underline{-16x^2 - 8x} \\
 & \qquad\qquad 40x + 20 \\
 & \qquad\qquad \underline{40x + 20} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

$p(x) = 2x^3 - 15x^2 + 32x + 20$

$= (2x+1)(x^2 - 8x + 20)$

where $a = -8$ and $b = 20$

Silver

b) One root when $2x+1=0$, $x=-\frac{1}{2}$

$\therefore$ Use discriminant for $x^2-8x+20$

$a=1$, $b=-8$, $c=20$

$$b^2-4ac = (-8)^2-4(1)(20)$$
$$= 64-80$$
$$= -16<0$$

↳ indicates no real roots $\therefore$ $p(x)=0$ has exactly one root

c) $2x^3-15x^2+32x+20 = (2x+1)(x^2-8x+20)$

$2x+1>0$ when $x>-\frac{1}{2}$ $\therefore$ $2x+1>0$ for all real positive values of x

Complete the square for $x^2-8x+20$:

$x^2-8x+20$

$(x-4)^2-16+20$

$(x-4)^2+4$

$(x-4)^2 \geq 0$ for all real values of x

$\therefore$ $(x-4)^2+4 \geq 4$ for all real values of x and $(2x+1)(x^2-8x+20)>0$ for all real positive values of x

Gold

$p(x) = 9x^3 - 6x^2 - 20x - 8$

a) Using factor theorem:

$f(1) = 9(1)^3 - 6(1)^2 - 20(1) - 8 = -25$ ✗

$f(-1) = 9(-1)^3 - 6(-1)^2 - 20(-1) - 8 = -3$ ✗

$f(2) = 9(2)^3 - 6(2)^2 - 20(2) - 8 = 0$ ✓ so factor is $(x-2)$

$$
\begin{array}{r|l}
 & 9x^2 + 12x + 4 \\ \hline
x - 2 & 9x^3 - 6x^2 - 20x - 8 \\
 & \underline{9x^3 - 18x^2} \\
 & \quad 12x^2 - 20x \\
 & \quad \underline{12x^2 - 24x} \\
 & \qquad 4x - 8 \\
 & \qquad \underline{4x - 8} \\
 & \qquad\quad 0
\end{array}
$$

$p(x) = 9x^3 - 6x^2 - 20x - 8$

$= (x-2)(9x^2 + 12x + 4)$

where $a = 9$, $b = 12$, $c = 4$, and $d = -2$

b) $p(x) = (x-2)(9x^2 + 12x + 4)$

One real root is when $x = 2$

$\therefore$ Use discriminant for $9x^2 + 12x + 4$

$a = 9$, $b = 12$, $c = 4$

$b^2 - 4ac = (12)^2 - 4(9)(4)$

$= 144 - 144$

$= 0$

↳ indicates one real root

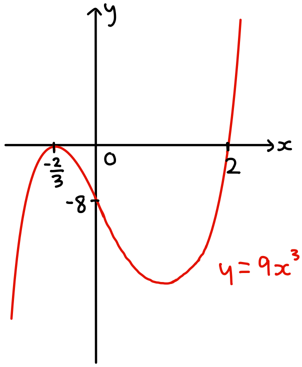
$\therefore$ $p(x) = 0$ has exactly two real roots

Gold

c) Curve with $y = 9x^3 - 6x^2 - 20x - 8$

$$= (x-2)(9x^2 + 12x + 4)$$
$$= (x-2)(3x+2)^2$$

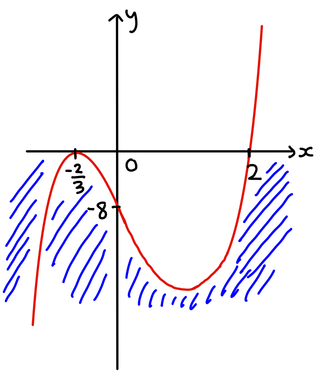
$x = 2$ or $x = -\frac{2}{3}$



d) $9x^3 - 6x^2 < 20x + 8$

$9x^3 - 6x^2 - 20x - 8 < 0$

$(x-2)(3x+2)^2 < 0$



$x < -\frac{2}{3}$ and $-\frac{2}{3} < x < 2$

$\therefore$ in set notation

$\left\{x : x < -\frac{2}{3}\right\}$ and $\left\{x : -\frac{2}{3} < x < 2\right\}$

Problem Solving: Set B

Bronze

$f(x) = x^3 - x^2 + px + q$

a) $(x+1)$ is a factor

$\therefore\ f(-1) = (-1)^3 - (-1)^2 + p(-1) + q = 0$

$= -1 - 1 - p + q = 0$

$= -2 - p + q = 0$

$= \quad -p + q = 2$

$= \quad q - p = 2$

b) $(x+3)$ is a factor

$\therefore\ f(-3) = (-3)^3 - (-3)^2 + p(-3) + q = 0$

$= -27 - 9 - 3p + q = 0$

$= -36 - 3p + q = 0$

$= \quad -3p + q = 36$

$= \quad q - 3p = 36$

c) $q - p = 2$

$q - 3p = 36$

Simultaneous equation:

$q = 2 + p$

$2 + p - 3p = 36$

$-2p = 34$

$p = -17$

$q = 2 + (-17)$

$q = -15$

$p = -17$ and $q = -15$

Bronze

d) $x^3 - x^2 + px + q$

$\hookrightarrow x^3 - x^2 - 17x - 15$

factorise to get:

$(x+1)(x+3)(x-5)$

$\therefore f(x) = (x+1)(x+3)(x-5)$

Silver

$f(x) = 2x^3 - x^2 + px + q$

a) $(x+2)$ is a factor

$\therefore f(-2) = 2(-2)^3 - (-2)^2 + p(-2) + q = 0$

$= -16 - 4 - 2p + q = 0$

$= -20 - 2p + q = 0$

$= q - 2p - 20 = 0$

b) $(x-3)$ is a factor

$\therefore f(3) = 2(3)^3 - (3)^2 + p(3) + q = 0$

$= 54 - 9 + 3p + q = 0$

$= 45 + 3p + q = 0$

$= q + 3p + 45 = 0$

$q - 2p - 20 = 0$

$q + 3p + 45 = 0$

$q = 20 + 2p$

$20 + 2p + 3p + 45 = 0$

$65 = -5p$

$-13 = p$

$q = 20 + 2(-13)$

$q = -6$

$p = -13$ and $q = -6$

Silver

c) $2x^3 - x^2 + px + q$

$\hookrightarrow 2x^3 - x^2 - 13x - 6$

factorise to get:

$(x+2)(x-3)(2x+1)$

$\therefore f(x) = (x+2)(x-3)(2x+1)$

Gold

$f(x) = x^3 + (p+4)x^2 + 8x + q$

a) $(x-2)$ is a factor

$\therefore f(2) = (2)^3 + (p+4)2^2 + 8(2) + q = 0$

$= 8 + 4p + 16 + 16 + q = 0$

$= 40 + 4p + q = 0$

$= 4p + q + 40 = 0$

b) $(x+p)$ is a factor

$\therefore f(-p) = (-p)^3 + (p+4)(-p)^2 + 8(-p) + q = 0$

$= -p^3 + p^3 + 4p^2 - 8p + q = 0$

$= 4p^2 - 8p + q = 0$

c) $4p + q + 40 = 0$

$4p^2 - 8p + q = 0$

$q = -40 - 4p$

$4p^2 - 8p + (-40 - 4p) = 0$

$4p^2 - 8p - 40 - 4p = 0$

$4p^2 - 12p - 40 = 0$

$p = 5$ or $p = -2$

$p > 0 \therefore p = 5$

$q = -40 - 4(5)$

$q = -60$

$p = 5$ and $q = -60$

Gold

d) $x^3 + (p+4)x^2 + 8x + q$

$\hookrightarrow x^3 + (5+4)x^2 + 8x - 60$

$x^3 + 9x^2 + 8x - 60$

factorise to get:

$(x-2)(x+5)(x+6)$

$\therefore f(x) = (x-2)(x+5)(x+6)$