

## Chapter 3 - Sequences and Series

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Personal notes:



## 3.1 - Arithmetic sequences

### Types of Sequences

2, 5, 8, 11, 14, ...

3, 6, 12, 24, 48, ...

1, 1, 2, 3, 5, 8, ...

### Terminologies and meaning

- $n$
- $u_n$

2, 5, 8, 11, 14, ...

### Arithmetic Sequence

- An arithmetic sequence is one...  
 $n^{\text{th}}$  term of arithmetic sequence:  
$$u_n = a + (n - 1)d$$

### Example

The  $n$ th term of an arithmetic sequence is  $u_n = 55 - 2n$ .

- Write down the first 3 terms of the sequence.
- Find the first term in the sequence that is negative.



## 3.1 - Arithmetic sequences

### Example

Find the  $n$ th term of each arithmetic sequence.

- a. 6, 20, 34, 48, 62
- b. 101, 94, 87, 80, 73

### Practice

**1** For each sequence:

- i** write down the first 4 terms of the sequence
- ii** write down  $a$  and  $d$ .

**a**  $u_n = 5n + 2$

**b**  $u_n = 9 - 2n$

**2** Find the  $n$ th terms and the 10th terms in the following arithmetic progressions:

**a** 5, 7, 9, 11, ...

**b** 5, 8, 11, 14, ...

### Example

A sequence is generated by the formula  $u_n = an + b$  where  $a$  and  $b$  are constants to be found.

Given that  $u_3 = 5$  and  $u_8 = 20$ , find the values of the constants  $a$  and  $b$ .



## 3.1 - Arithmetic sequences

### Example

For which values of  $x$  would the expressions  $-8$ ,  $x^2$  and  $17x$  form the first three terms of an arithmetic sequence.

### Exam Practice

#### Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for  $A$  minutes on day 1, where  $A$  is a constant.

She will then increase her running time by  $(d + 1)$  minutes each day, where  $d$  is a constant.

(a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for  $(A - 13)$  minutes on day 1.

She will then increase her running time by  $(2d - 1)$  minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of  $d$ .

(3)



## 3.2 - Arithmetic series

### Arithmetic Series

- Series is a \_\_\_ of terms in a sequence.
- Arithmetic Series (sum of first  $n$  terms):

#### Example ("Proof" - Numerical Version)

Prove the sum of the first 5 terms of this sequence: 2, 5, 8, 11, 14, 17... to be 40.

#### Example (Exam-able)

Prove that the sum of the first  $n$  terms of an arithmetic series is  $\frac{n}{2}(2a + (n - 1)d)$ .



## 3.2 - Arithmetic series

### Example

Find the sum of the first 30 terms of the following arithmetic sequences...

- a.  $2 + 5 + 8 + 11 + 14 + \dots$
- b.  $100 + 98 + 96 + 94 + \dots$
- c.  $p + 2p + 3p + \dots$

### Example

Find the least number of terms required for the sum of  $4 + 9 + 14 + 19 + \dots$  to exceed 2000.



## 3.2 - Arithmetic series

### Practice

1 Find the sums of the following series.

a  $3 + 7 + 11 + 14 + \dots$  (20 terms)

b  $2 + 6 + 10 + 14 + \dots$  (15 terms)

c  $30 + 27 + 24 + 21 + \dots$  (40 terms)

d  $5 + 1 + -3 + -7 + \dots$  (14 terms)

2 Find how many terms of the following series are needed to make the given sums.

a  $5 + 8 + 11 + 14 + \dots = 670$

b  $3 + 8 + 13 + 18 + \dots = 1575$

### Exam Practice

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is  $£P$ .

Salary increases by  $£(2T)$  each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is  $£(P + 1800)$ .

Salary increases by  $£T$  each year, forming an arithmetic sequence.

(a) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(b) Find the value of  $T$ .

(4)

For this value of  $T$ , the salary in Year 10 under Salary Scheme 2 is  $£29\,850$ .

(c) Find the value of  $P$ .

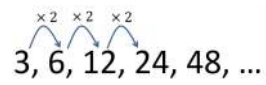
(3)



## 3.3 - Geometric sequences

### Notes

- A geometric sequence is one...



### Quickfire common ratio

Identify the common ratio  $r$ :

- 1 1, 2, 4, 8, 16, 32, ...
- 2 27, 18, 12, 8, ...
- 3 10, 5, 2.5, 1.25, ...
- 4 5, -5, 5, -5, 5, -5, ...
- 5  $x, -2x^2, 4x^3$
- 6  $1, p, p^2, p^3, \dots$
- 7 4, -1, 0.25, -0.0625, ...

### Geometric Sequence

$n^{\text{th}}$  term of a geometric sequence:

$$u_n =$$



## 3.3 - Geometric sequences

### Example

Determine the  $n^{\text{th}}$  term and the  $10^{\text{th}}$  term of the following sequences:

- 3, 6, 12, 24, ...
- 40, -20, 10, -5, ...

### Example

The 2nd term of a geometric sequence is 4 and the 4th term is 8. Given that the common ratio is positive, find the exact value of the 11th term in the sequence.

### Example

The numbers 3,  $x$  and  $(x + 6)$  form the first three terms of a geometric sequence with all positive terms. Find

- The possible values of  $x$ ,
- the 10th term of the sequence



## 3.3 - Geometric sequences

### Example

What is the first term in the geometric progression 3, 6, 12, 24, ... to exceed 1 million?

### Practice

All the terms in a geometric sequence are positive.

The third term of the sequence is 20 and the fifth term 80. What is the 20<sup>th</sup> term?

### Practice

The second, third and fourth term of a geometric sequence are the following:

$$x, \quad x + 6, \quad 5x - 6$$

- Determine the possible values of  $x$ .
- Given the common ratio is positive, find the common ratio.
- Hence determine the possible values for the first term of the sequence.



## 3.4 - Geometric series

### Notes

Geometric series formula:

- $S_n =$

### Proof (Exam-able)

A geometric series has first term  $a$  and common difference  $r$ . Prove that the sum of the first  $n$  terms of this series is given by  $S_n = \frac{a(1-r^n)}{1-r}$

### Example

Find the sums of the following geometric series.

**a**  $2 + 6 + 18 + 54 + \dots$  (for 10 terms)

**b**  $1024 - 512 + 256 - 128 + \dots + 1$

### Practice

Find the sum of the first 10 terms:

$3, 6, 12, 24, 48, \dots$

**BF MATHS**

## 3.4 - Geometric series

### Practice

Find the sum of the first 10 terms

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

### Example

Find the least value of  $n$  such that the sum of  $1 + 2 + 4 + 8 + \dots$  to  $n$  terms would exceed 2 000 000.

### Exam Practice

#### Edexcel C2 June 2011 Q6

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

- (a) the common ratio. (2)
- (b) the first term. (2)
- (d) the smallest value of  $n$  for which the sum of the first  $n$  terms of the series exceeds 1000. (4)



## 3.5 - Sum to infinity

### Divergent and Convergent

#### Notes

A sequence \_\_\_\_\_ if it tends to (gets closer and closer to) a certain value and \_\_\_\_\_ if it does not tend to a certain value. For example,

(a) 8, 4, 2, 1, 0.5, 0.25, ... \_\_\_\_\_

(b)  $9, 9\frac{1}{2}, 9\frac{3}{4}, 9\frac{7}{8}, \dots$  \_\_\_\_\_

(c) 5, 8, 11, 14, ... \_\_\_\_\_

A series \_\_\_\_\_ if the infinity sum of the terms adds up to a certain value and \_\_\_\_\_ if the infinity sum does not add up to a certain value. For example,

(a)  $1 + 2 + 3 + 4 + 5 + \dots$ , \_\_\_\_\_

(b)  $1 - 2 + 3 - 4 + 5 - 6 + \dots$ , \_\_\_\_\_

(c)  $1 + 0.5 + 0.25 + 0.125 + \dots$ , \_\_\_\_\_

#### Understanding the difference

$1 + 0.5 + 0.25 + 0.125 + \dots$  ← Why did this infinite sum converge (to 2)...

$1 + 2 + 4 + 8 + 16 + \dots$  ← ...but this diverge to infinity?

\*\* A geometric series is convergent if \_\_\_\_\_



## 3.5 - Sum to infinity

The formula

### Examples

Find the sum to infinity for each of the following geometric series:

a)  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

b)  $27 + (-9) + 3 + (-1) + \dots$

c)  $p + p^2 + p^3 + \dots$  (where  $-1 < p < 1$ )

### Example

The fourth term of a geometric series is 1.08 and the seventh term is 0.233 28.

**a** Show that this series is convergent.

**b** Find the sum to infinity of the series.



## 3.5 - Sum to infinity

### Example

For a geometric series with first term  $a$  and common ratio  $r$ ,  $S_4 = 15$  and  $S_\infty = 16$ .

- a Find the possible values of  $r$ .
- b Given that all the terms in the series are positive, find the value of  $a$ .

### Exam Practice

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

- (a) the common ratio, (2)
- (b) the first term, (2)
- (c) the sum to infinity, (2)
- (d) the smallest value of  $n$  for which the sum of the first  $n$  terms of the series exceeds 1000. (4)



## 3.6 - Sigma notation

### Notes

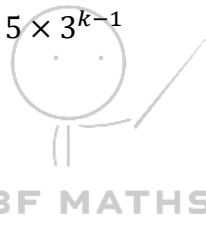
What does each bit of this expression mean?

$$\sum_{r=1}^5 (2r + 1)$$

### Example

	First 5 terms	Values of a, n, d or r	Final result
$\sum_{n=1}^7 3n$			

$$\sum_{k=5}^{15} (10 - 2k)$$

$$\sum_{k=1}^{12} 5 \times 3^{k-1}$$


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## 3.6 - Sigma notation

### Practice

Evaluate

$$\sum_{r=10}^{30} (7 + 2r). \quad (4)$$



## 3.7 - Recurrence Relations

### Notes

- Position-to-term sequence
  
- Term-to-term sequence (aka recurrence relation)

\*\*\*Important Note: With recurrence relation questions, the **sequence will likely not be arithmetic nor geometric**. So your previous  $u_n$  and  $S_n$  formulae do not apply.

### Example

Find the first four terms of the following sequences.

**a**  $u_{n+1} = u_n + 4, u_1 = 7$

**b**  $u_{n+1} = u_n + 4, u_1 = 5$



## 3.7 - Recurrence Relations

### Example

A sequence  $a_1, a_2, a_3, \dots$  is defined by

$$a_1 = p$$

$$a_{n+1} = (a_n)^2 - 1, n \geq 1$$

where  $p < 0$ .

**a** Show that  $a_3 = p^4 - 2p^2$ .

**b** Given that  $a_2 = 0$ , find the value of  $p$ .

**c** Find  $\sum_{r=1}^{200} a_r$

**d** Write down the value of  $a_{199}$

Edexcel C1 Jan 2012

### Exam Practice

**4.** A sequence  $x_1, x_2, x_3, \dots$  is defined by

$$x_1 = 1,$$

$$x_{n+1} = ax_n + 5, \quad n \geq 1,$$

where  $a$  is a constant.

(a) Write down an expression for  $x_2$  in terms of  $a$ .

(1)

(b) Show that  $x_3 = a^2 + 5a + 5$ .

(2)

Given that  $x_3 = 41$

(c) find the possible values of  $a$ .

(3)



## 3.7 - Recurrence Relations

### Notes

- A sequence is *increasing* if
- A sequence is *decreasing* if
- A sequence is *periodic* if the terms repeat in a cycle.  
- the order of the sequence is an integer such that

### Examples

#### Example

Determine if each of the following sequences is increasing, decreasing or periodic. If it is periodic, write down its order.

a)  $u_{n+1} = u_n + 3, u_1 = 7$

b)  $u_{n+1} = (u_n)^2, u_1 = \frac{1}{2}$

c)  $u_n = \sin(90n^\circ)$



## 3.8 - Modelling with series

You will usually need to use the formulae for arithmetic and geometric sequences. In problem solving questions you will need to figure out which sequence is the right one to use.

### Recall

You need to know the difference between the words **sequence** and **series**.

**Sequence:** 3, 5, 7, 9, 11, 13 is a sequence with six terms.

The sum of these terms is called a **series** and can be found using the formula for arithmetic series.

i.e.  $3 + 5 + 7 + 9 + 11 + 13$

**Arithmetic Sequence:**  $u_n = a + (n - 1)d$

**Geometric Sequence:**  $u_n = ar^{n-1}$

**Arithmetic Series:**  $S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n - 1)d]$

**Geometric Series:**  $S_n = \frac{a(1-r^n)}{1-r}$

All given in the formula booklet!

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Arithmetic Series	Geometric Series
<ul style="list-style-type: none"><li>• <math>S_n</math> is the sum of the first <math>n</math> terms</li><li>• <math>n</math> is the number of terms</li><li>• <math>a</math> is the first term</li><li>• <math>l</math> is the last term</li><li>• <math>d</math> is the common difference (the difference between consecutive terms)</li></ul>	<ul style="list-style-type: none"><li>• <math>S_n</math>, <math>a</math> and <math>n</math> are as above</li><li>• <math>r</math> is the common ratio (the number you multiply a term by to find the next term)</li><li>• <math>S_\infty</math> is the sum of an infinite series</li></ul>

\*\*In short, if you are looking to find the next term -> **Sequence**

if you are looking to find the sum after certain "years" -> **Series**



## 3.8 - Modelling with series

### Example

Fatou started a business selling artworks online and George started a business selling second hand books online.

Fatou made a profit of £100 in the first month. He expects his profit to increase by 10% in each following month, so the profit will be £133.10 in the fourth month.

George made a profit of £150 in the first month. He thinks his profit will increase by £30 in each following month, so the profit will be £240 in the fourth month.

- Find an expression of the total profit made by Fatou in the first  $n$  months.
- Find an expression of the total profit made by George in the first  $n$  months.
- Compare and comment on both Fatou's and George's models for their long-term total profits.

### Example

The value of a new house is £430 000. It is expected to increase in value by 12% each year.

- Find an expression for the value of the house after  $n$  years.
- Use logarithms to forecast when the value of the house exceeds £1 million, to the nearest year.
- Comment on why this forecast may not be reliable.



## 3.8 - Modelling with series

### Example

Bruce starts a new company. In year 1, his profits will be £20,000. He predicts his profits to increase by £5,000 each year, so that his profits in year 2 are modelled to be £25,000, in year 3 £30,000, and so on. He predicts this will continue until he reaches annual profits of £100,000. He then models his annual profits to remain at £100,000.

- Calculate the profits for Bruce's business in the first 20 years.
- State one reason why this may not be a suitable model.
- Bruce's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

### Exam Practice

#### Edexcel C2 Jan 2013 Q3

A company predicts a yearly profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%. The predicted yearly profit forms a geometric sequence with common ratio 1.05.

- Show that the predicted profit in the year 2016 is £138 915. (1)
- Find the first year in which the yearly predicted profit exceeds £200 000. (5)
- Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound. (3)

