

Chapter 9 - Differentiation



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End of topic task:

Write down a revision summary on the next page

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Personal Notes



9.1 - Differentiating $\sin x$ and $\cos x$

Recall

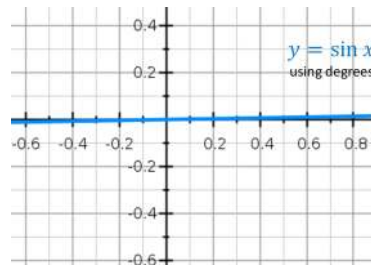
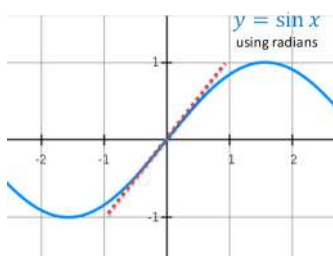
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$

Small Angle approximations

- $\sin \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$

****Always use radians when differentiating trig functions!**

Why?



Example

Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume that as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$.



9.1 - Differentiating $\sin x$ and $\cos x$

Practice (Ex. 9A Q10)

E/P 10 Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume the formula for $\sin(A + B)$ and that as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$.

(5 marks)

Differentiating $\sin kx$ and $\cos kx$

- $\frac{d}{dx}(\sin kx) = k \cos kx$
- $\frac{d}{dx}(\cos kx) = -k \sin kx$

Example

Find $\frac{dy}{dx}$ given that:

a $y = \sin 2x$

b $y = \cos 5x$

c $y = 3 \cos x + 2 \sin 4x$



9.1 - Differentiating $\sin x$ and $\cos x$

Practice

a) $\frac{d}{dx}(3 \sin 5x) =$

b) $\frac{d}{dx}(4 \cos 3x) =$

c) $\frac{d}{dx}\left(-\frac{1}{2} \sin x\right) =$

d) $\frac{d}{dx}\left(-\frac{2}{3} \cos \frac{1}{2}x\right) =$

Example

A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \leq x \leq \pi$.



9.2 - Differentiating exponentials and logarithms

Notes

Example

Show that the derivative of a^x is $a^x \ln a$.

Practice

$$\frac{d}{dx}(3^x) =$$

$$\frac{d}{dx}(x^3) =$$

$$\frac{d}{dx}(\ln(3x)) =$$

$$\frac{d}{dx}(3^{2x}) =$$

$$\frac{d}{dx}(2^{3x}) =$$

$$\frac{d}{dx}(5 \ln x) =$$

$$\frac{d}{dx}\left(e^{\frac{1}{2}x}\right) =$$

$$\frac{d}{dx}(5 \ln(2x)) =$$

$$\frac{d}{dt}(9^t) =$$

$$\frac{d}{dx}(5(4^x)) =$$

$$\frac{d}{dx}(x^4) =$$

9.2 - Differentiating exponentials and logarithms

Example

Find $\frac{dy}{dx}$ given that:

a $y = e^{3x} + 2^{3x}$

b $y = \ln(x^3) + \ln 7x$

c $y = \frac{2 - 3e^{7x}}{4e^{3x}}$

Example (head-scratcher)

A child has headlice and his parents treat it using a special shampoo. The population P of headlice after t days can be modelled using $P = 460(3^{-2t})$

- Determine how many days have elapsed before the child has 20 headlice left.
- Determine the rate of change of headlice after 3 days.



Differentiating Combinations of Functions

Functions can interact in different ways...

1 Composite Function

i.e. of form $y = f(g(x))$

$$y = \sqrt{1 + 3x}$$

The 'outer' function here is the $\sqrt{\quad}$ and the inner function the $1 + 3x$.

i.e. $f(x) = \sqrt{x}$ and $g(x) = 1 + 3x$

How to
differentiate



The Chain Rule (Chp 9.3)

2 Product of Two Functions

i.e. of form $y = f(x)g(x)$

$$y = x \sin 2x$$



The Product Rule (Chp 9.4)

3 Division (i.e. "Quotient") of Two Functions

i.e. of form $y = \frac{f(x)}{g(x)}$

$$y = \frac{\ln x}{x}$$



The Quotient Rule (Chp 9.5)



9.3 - Chain Rule

Chain Rule is used to differentiate composite functions.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where....

Example

Given that $y = (3x^4 + x)^5$, find $\frac{dy}{dx}$ using the chain rule.

Practice

Identify the expression of u in each of the questions and hence to differentiate

$$y = (x^2 + 1)^3$$

$$y = (\ln x)^3$$

$$y = e^{x^2+x}$$

$$y = (2^x + 1)^2$$

$$y = \ln(\sin x)$$

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9.3 - Chain Rule

Example

Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$ at (4,9)

Example

Find the value of $\frac{dy}{dx}$ at the point (2, 1) on the curve with equation $y^3 + y = x$.

Exam Question

C3 June 2011 Q1a

Differentiate with respect to x

(a) $\ln(x^2 + 3x + 5)$, (2)

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9.4 - Product Rule

- The product rule is used when finding the derivative of a product of two functions.
- Product rule is:

Example (Product)

If $y = x^2 \sin x$, determine $\frac{dy}{dx}$

Example (Product + Chain)

Given that $f(x) = x^2\sqrt{3x-1}$, find $f'(x)$.



9.4 - Product Rule

Example

Given that $y = e^{4x} \sin^2 3x$, show that $\frac{dy}{dx} = e^{4x} \sin 3x (A \cos 3x + B \sin 3x)$, where A and B are constants to be determined.

Practice

Edexcel C3 Jan 2012 Q1a

Differentiate with respect to x , giving your answer in its simplest form,

(a) $x^2 \ln(3x)$,

(4)



9.5 - Quotient Rule

- Quotient Rule is...

Example

Given that $y = \frac{x}{2x+5}$ find $\frac{dy}{dx}$

Example

A curve C with equation $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$,
has a stationary point at P . Find the coordinates
of P . Give your answer to 3 significant figures.



9.5 - Quotient Rule

Exam Practice

Edexcel C3 June 2012 Q3

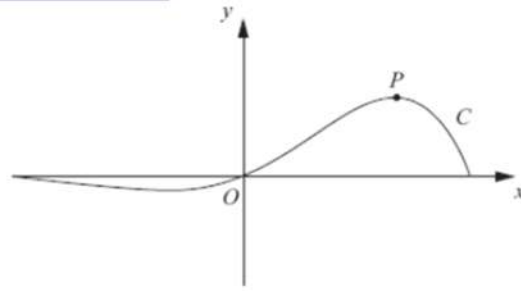


Figure 1

Figure 1 shows a sketch of the curve C which has equation

$$y = e^{x^3} \sin 3x, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}.$$

- (a) Find the x -coordinate of the turning point P on C , for which $x > 0$.
Give your answer as a multiple of π . (6)
- (b) Find an equation of the normal to C at the point where $x = 0$. (3)



Summary (Chain, Product, Quotient)



9.6 - Differentiating Trigonometric Functions

Example

Differentiate $y = \tan x$

Example

Differentiate $y = \sec x$

Given in Formula Booklet

■ If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k \operatorname{cosec} kx \cot kx$

■ If $y = \sec kx$, then $\frac{dy}{dx} = k \sec kx \tan kx$

■ If $y = \cot kx$, then $\frac{dy}{dx} = -k \operatorname{cosec}^2 kx$

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9.6 - Differentiating Trigonometric Functions

Example

Differentiate: **a** $y = \frac{\operatorname{cosec} 2x}{x^2}$ **b** $y = \sec^3 x$

Practice

Edexcel C3 June 2013(R) Q5b

(b) Show that $\frac{d}{dx}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

(3)



9.6 - Differentiating Trigonometric Functions

Often in exam questions, you will be given x in terms of y , but want to find $\frac{dy}{dx}$ in terms of x .

The key is to make use of an appropriate trig identity, e.g:

$$\sin^2 x + \cos^2 x \equiv 1 \quad 1 + \tan^2 x \equiv \sec^2 x$$

Example

Given that $x = \tan y$, express $\frac{dy}{dx}$ in terms of x

Example

Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$



9.7 - Parametric Differentiation

- If x and y are given as functions of a parameter t , then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example

Find the gradient at the point P where $t = 2$, on the curve given parametrically by

$$x = t^3 + t, \quad y = t^2 + 1, \quad t \in \mathbb{R}$$

Example

Find the equation of the normal at the point P where $\theta = \frac{\pi}{6}$, to the curve with parametric equations $x = 3 \sin \theta$, $y = 5 \cos \theta$.



9.8 - Implicit Differentiation

Notes

- An equation in the form $y = f(x)$ is given **explicitly**.
- Equations which involve both x and y are called **implicit** equations.

Explicit Differentiation (aka normal differentiation)

To differentiate implicitly :

- Differentiate **each side** of the equation (use ~~union method~~ chain rule if necessary)
 - When differentiating a function of y with respect to x , differentiate as if it is x but **slap** a $\frac{dy}{dx}$ **on the end**.
- i.e. $\frac{d}{dx}(f(y)) = f'(y) \frac{dy}{dx}$

Examples

1. $\frac{d}{dx}(y^2)$

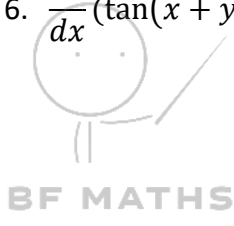
2. $\frac{d}{dx}(\sin y)$

3. $\frac{d}{dx}(e^y)$

4. $\frac{d}{dx}(xy)$

5. $\frac{d}{dx}(e^{x^2y})$

6. $\frac{d}{dx}(\tan(x + y))$



9.8 - Implicit Differentiation

Exam Tips:

$\frac{d}{dx}(xy)$ is extremely common, which equals to $x \frac{dy}{dx} + y$ by Product Rule, suggest to remember it.

Example

Find $\frac{dy}{dx}$ in terms of x and y where $x^3 + x + y^3 + 3y = 6$.

Example

Find the value of $\frac{dy}{dx}$ at the point $(1, 1)$ where $e^{2x} \ln y = x + y - 2$.



9.8 - Implicit Differentiation

Exam Practice

C4 Jan 2008 Q5

A curve is described by the equation

$$x^3 - 4y^2 = 12xy.$$

- (a) Find the coordinates of the two points on the curve where $x = -8$. (3)
- (b) Find the gradient of the curve at each of these points. (6)

Exam Practice

C4 June 2014(R) Q3

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

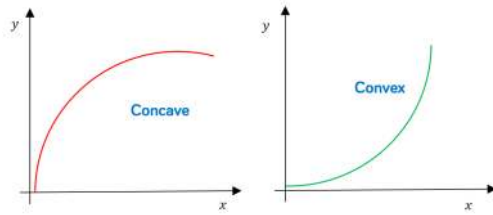
- (a) Find $\frac{dy}{dx}$ in terms of x and y , fully simplifying your answer. (5)
- (b) Find the values of y for which $\frac{dy}{dx} = 0$. (5)



9.9 - Using Second Derivative $f''(x)$

Notes

We can use second derivative to determine if a curve is **concave** or **convex** on a given domain. (or informally, bending one way to bending the other)



- $f(x)$ is **concave** when $f''(x) \leq 0$
- $f(x)$ is **convex** when $f''(x) \geq 0$
- The point which changes from concave to convex (vice versa) is called **point of inflection**.
 - To show that, the sign of $f''(x)$ changes.
 - To find the coordinates, $f''(x) = 0$

Example

Find the interval on which the function $f(x) = x^3 + 4x + 3$ is concave.

Example

Show that the function $f(x) = e^{2x} + x^2$ is convex for all real values of x .



9.9 - Using Second Derivative

Example

The curve C has equation $y = x^3 - 2x^2 - 4x + 5$.

- a Show that C is concave on the interval $[-2, 0]$ and convex on the interval $[1, 3]$.
- b Find the coordinates of the point of inflection.



9.10(1) - Rates of Change

Notes

Example

Given that the area of a circle $A \text{ cm}^2$ is related to its radius $r \text{ cm}$ by the formula $A = \pi r^2$, and that the rate of change of its radius in cm s^{-1} is given by $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 3$.

Example

The volume of a hemisphere $V \text{ cm}^3$ is related to its radius $r \text{ cm}$ by the formula $V = \frac{2}{3}\pi r^3$ and the total surface area $S \text{ cm}^2$ is given by the formula $S = \pi r^2 + 2\pi r^2 = 3\pi r^2$. Given that the rate of increase of volume, in $\text{cm}^3 \text{ s}^{-1}$, $\frac{dV}{dt} = 6$, find the rate of increase of surface area $\frac{dS}{dt}$



9.10(1) - Rates of Change

Example

Edexcel C4 June 2008 Q3

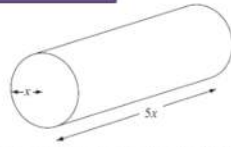


Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of $0.032 \text{ cm}^2 \text{ s}^{-1}$.

- (a) Find $\frac{dx}{dt}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures. (4)
- (b) Find the rate of increase of the volume of the rod when $x = 2$. (4)



9.10(1) - Rates of Change

Practice

June 2012 Q2

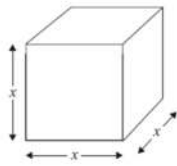


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm³.

(a) Show that $\frac{dV}{dx} = 3x^2$.

(1)

Given that the volume, V cm³, increases at a constant rate of 0.048 cm³ s⁻¹,

(b) find $\frac{dx}{dt}$ when $x = 8$,

(2)

(c) find the rate of increase of the total surface area of the cube, in cm² s⁻¹, when $x = 8$.

(3)



9.10(2) - Constructing Differential Equation

- An equation which involves a rate of change is called a **differential equation**.
- You are required to know how to construct a differential equation (harder part)
- And learn to solve it through integration (Chapter 11)

Example

In the decay of radioactive particles, the rate at which particles decay per minute, t , is proportional to the number of particles remaining, N . Write down a differential equation for the rate of change of the number of particles per minute.

Example

The head of a snowman of radius R cm loses volume by evaporation at a rate proportional to its surface area. Assuming that the head is spherical, that the volume of a sphere is $\frac{4}{3}\pi R^3$ cm³ and that the surface is $4\pi R^2$ cm², write down a differential equation for the rate of change of radius of the snowman's head.



9.10(2) - Constructing Differential Equation

Practice Q1

A population is growing at a rate which is proportional to the size of the population.
Write down a differential equation for the growth of the population.

Practice Q2

Liquid is pouring into a container at a constant rate of $30 \text{ cm}^3 \text{ s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V \text{ cm}^3 \text{ s}^{-1}$, where $V \text{ cm}^3$ is the volume of the liquid in the container at that time.

Show that $-15 \frac{dV}{dt} = 2V - 450$.

Practice Q3

An electrically-charged body loses its charge, Q coulombs, at a rate, measured in coulombs per second, proportional to the charge Q .

Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge.

Practice Q4

The ice on a pond has a thickness x mm at a time t hours after the start of freezing. The rate of increase of x is inversely proportional to the square of x .

Write down a differential equation in terms of x and t .



9.10(2) - Constructing Differential Equation

Practice Q5

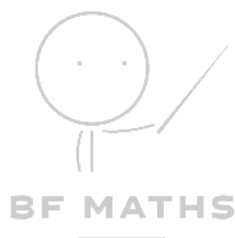
The radius of a circle is increasing at a constant rate of 0.4 cm per second.

- Find $\frac{dC}{dt}$, where C is the circumference of the circle, and interpret this value in the context of the model.
- Find the rate at which the area of the circle is increasing when the radius is 10 cm.
- Find the radius of the circle when its area is increasing at the rate of 20 cm^2 per second.

Practice Q6

Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t > 0$, the volume of fluid remaining in the tank is $V \text{ m}^3$. The rate at which the fluid flows in $\text{m}^3 \text{ min}^{-1}$ is proportional to the square root of V .

Show that the depth, h metres, of fluid in the tank satisfies the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant.



9.10(2) - Constructing Differential Equation

Practice Q7

An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of $6 \text{ cm}^3 \text{ s}^{-1}$.

Given that the angle of the cone between the slanting edge and the vertical is 30° , show that the volume of the salt is $\frac{1}{9}\pi h^3$, where h is the height of salt at time t seconds. Show that the rate of change of the height of the salt in the funnel is inversely proportional to h^2 . Write down a differential equation relating h and t .

